ORTHOGONAL POLYNOMIALS AND THE TODA EQUATIONS

The type of results that appeal to me and I try to include them in my classes is `unexpected relations` among subjects that apriori seem unrelated. This connections might be clear to the expert, but the students seem to enjoy them. Here is one of them:

Start with a sequence of polynomials $\{p_n\}$ with $\deg p_n = n$. Then, for any fixed $k \in \mathbb{N}$, the set $\{p_0,\, p_1, \cdots, p_k\}$ is a basis for the vector space $\mathbb{V}_k$ of polynomials of degree at most k. It is often convenient to assume that this basis is `orthonormal` with respect to an inner product defined on $\mathbb{V}_k$. This can be given in terms of a `positive measure` by the rule

$$\langle f, g\rangle = \int_I f(x)g(x)d\mu(x).$$

and the condition of orthonormality is simply that

$$\langle p_k, p_j\rangle = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j \end{cases}$$

For readers unaware of measures just think of this inner product as given by a `weight` w in the form

$$\langle f, g\rangle = \int_I f(x)g(x)w(x)\, dx.$$

The `Gram-Schmidt process` takes a basis for a vector space with an inner product and returns an orthonormal basis. This is simply the way you take 3 linearly independent vectors in $\mathbb{R}^3$ and make them perpendicular to each other by use of projections. The text [1] contains a very nice description of these ideas.

Now we start with $\{p_n\}$, a sequence of orthonormal polynomials with respect to a positive measure μ, and assume $\deg p_n = n$. Then

$$\int_{\mathbb{R}} p_n(x)p_m(x)\, d\mu(x) = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Since $xp_n(x)$ is a polynomial of degree $n+1$, one has

$$xp_n(x) = \sum_{j=1}^{n+1} u_{j,n}p_j(x), \tag{1}$$

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for some coefficients $u_{j,n}$. Multiply by $p_r(x)$, in the range $1 \leq r \leq n+1$, and integrate to obtain

$$\int x p_n(x) p_r(x)\, d\mu(x) = \sum_{j=1}^{n+1} u_{j,n} \int p_j(x) p_r(x)\, d\mu(x) = u_{r,n},$$

by orthogonality. Then

$$u_{r,n} = \int x p_n(x) p_r(x)\, d\mu(x) = \int p_n(x) \left(x p_r(x) \right)\, d\mu(x).$$

If $r+1 < n$, then the polynomial $x p_r(x)$ (being of degree $r+1$) is orthogonal to $p_n(x)$ and this implies $u_{r,n} = 0$. Then (1) reduces to

$$x p_n(x) = u_{n-1,n} p_{n-1}(x) + u_{n,n} p_n(x) + u_{n+1,n} p_{n+1}(x). \tag{2}$$

Denote $u_{n-1,n}$ by a_n and observe that

$$a_n = u_{n-1,n} = \int x p_n(x) p_{n-1}(x)\, d\mu(x). \tag{3}$$

Now

$$u_{n+1,n} = \int x p_n(x) p_{n+1}(x)\, d\mu(x) = \int x p_{n+1}(x) p_n(x)\, d\mu(x) = a_{n+1}. \tag{4}$$

Denote $b_n = u_{n,n}$ to obtain from (2)

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{5}$$

that is,

$$a_{n+1} p_{n+1}(x) + (b_n - x) p_n(x) + a_n p_{n-1}(x) = 0. \tag{6}$$

Therefore the three-term relation has been expressed in terms of two sequences $\{a_n\}$ and $\{b_n\}$.

Theorem 1. Let $\{p_n\}$ be a sequence of orthonormal polynomials with respect to a positive measure μ. Assume that all the moments of $d\mu$ are finite. Then there is a third order recurrence of the form

$$a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x),$$

where

$$a_n = \int x p_n(x) p_{n-1}(x)\, d\mu(x) \text{ and } b_n = \int x p_n^2(x)\, d\mu(x).$$

Let γ_n be the leading coefficient of p_n. Comparing the leading coefficients in (6) gives

$$a_{n+1} = \frac{\gamma_n}{\gamma_{n+1}}.$$

Introduce

$$P_n(x) = \frac{1}{\gamma_n} p_n(x),$$

the monic version of the polynomials $\{p_n\}$. Then the three-term recurrence becomes

$$x\gamma_n P_n(x) = a_{n+1}\gamma_{n+1}P_{n+1}(x) + b_n\gamma_n P_n(x) + a_n\gamma_{n-1}P_{n-1}(x).$$

Divide by γ_{n+1} to obtain the next statement.

Theorem 2. The monic polynomials $P_n(x)$ satisfy the recurrence

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x). \tag{7}$$

Now consider the sequence of orthonormal polynomials $\{p_n(x)\}$ constructed for a measure $d\mu_t = e^{xt}d\mu$. These satisfy a third order recurrence

$$xp_n(x,t) = a_{n+1}(t)p_{n+1}(x,t) + b_n(t)p_n(x,t) + a_n(t)p_{n-1}(x,t), \tag{8}$$

where now the coefficients depend on t. The next discussion appears in [3].

Recall that

$$a_n(t) = \int xp_n(x,t)p_{n-1}(x,t)d\mu_t(x) \text{ and } b_n(t) = \int xp_n^2(x,t)d\mu_t(x). \tag{9}$$

To simplify notation, we will drop the variables in $p_n(x,t)$ in (8) and write it as

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}. \tag{10}$$

Differentiate with respect to t gives

$$\begin{aligned} xp_n' &= a_{n+1}'p_{n+1} + b_n'p_n + a_n'p_{n-1} \\ &+ a_{n+1}p_{n+1}' + b_n p_n' + a_n p_{n-1}', \end{aligned} \tag{11}$$

where $'$ is derivative with respect to t. Observe that the degree of p_n' (in x) is at most the degree of p_n (the leading coefficients might disappear). Multiply (11) by p_{n+1} and integrate to produce

$$\begin{aligned} \int xp_n'p_{n+1}\,d\mu_t &= a_{n+1}'\int p_{n+1}^2\,d\mu_t + b_n'\int p_np_{n+1}\,d\mu_t \\ &+a_n'\int p_{n-1}p_{n+1}\,d\mu_t + a_{n+1}\int p_{n+1}'p_{n+1}\,d\mu_t \\ &+b_n\int p_n'p_{n+1}\,d\mu_t + a_n\int p_{n-1}'p_{n+1}\,d\mu_t. \end{aligned} \tag{12}$$

Now recall that the polynomials p_n are orthonormal, so the integral against any polynomial of lower degree vanishes. This reduces (12) to

$$\int xp_n'p_{n+1}\,d\mu_t = a_{n+1}' + a_{n+1}\int p_{n+1}p_{n+1}'\,d\mu_t. \tag{13}$$

Now multiply the relation

$$xp_{n+1} = a_{n+2}p_{n+2} + b_{n+1}p_{n+1} + a_{n+1}p_n \tag{14}$$

by p_n' and integrate. The non-vanishing terms are

$$\int xp_n'p_{n+1} = a_{n+1}\int p_np_n' \tag{15}$$

and replacing in (14) gives

$$a_{n+1}' = a_{n+1}\left[\int p_n p_n' - \int p_{n+1}p_{n+1}'\right]. \tag{16}$$

To simplify this relation, differentiate the normalization

$$\int p_n^2 d\mu_t = \int p_n^2 e^{xt}\, d\mu = 1, \tag{17}$$

with respect to t to produce

$$\int \left(xp_n^2 + 2p_n p_n'\right) e^{xt}\, d\mu = 0, \tag{18}$$

from which it follows that

$$b_n = \int xp_n^2\, d\mu_t = -2\int p_n p_n'\, d\mu_t. \tag{19}$$

Replace this in (16) to obtain

$$a_{n+1}' = a_{n+1}\left[-\tfrac{1}{2}b_n + \tfrac{1}{2}b_{n+1}\right]. \tag{20}$$

This equation is written as

$$a_n' = \tfrac{1}{2}a_n(b_n - b_{n-1}) \tag{21}$$

after shifting n to $n-1$. Multiply by a_n to produce the alternative form

$$\frac{d}{dt}a_n^2 = a_n^2\left(b_n - b_{n-1}\right). \tag{22}$$

Now multiply (11) by p_n and integrate. The non-vanishing terms give

$$\int xp_n p_n' = b_n' + a_{n+1}\int p_{n+1}' p_n + b_n \int p_n' p_n. \tag{23}$$

Then multiply (10) by p_n and integrate to derive

$$\int xp_n p_n'\, d\mu_t = b_n \int p_n p_n'\, d\mu_t + a_n \int p_n' p_{n-1}\, d\mu_t \tag{24}$$

from which one obtains

$$b_n' = a_n \int p_n' p_{n-1}\, d\mu_t - a_{n+1}\int p_{n+1}' p_n\, d\mu_t. \tag{25}$$

To simplify this identity, differentiate

$$\int p_n p_{n+1} e^{xt}\, d\mu(x) = 0 \tag{26}$$

to obtain

$$\int p_{n+1}' p_n + \int xp_{n+1}p_n = 0. \tag{27}$$

Therefore (9) gives

$$\int p_{n+1}' p_n\, d\mu_t = -a_{n+1}. \tag{28}$$

Replacing in (25) gives

$$\frac{db_n}{dt} = a_{n+1}^2 - a_n^2. \tag{29}$$

Theorem 3. Let $\{p_n\}$ be the sequence of orthonormal polynomials with respect to the measure $d\mu_t = e^{xt}d\mu$. These polynomials satisfy the recurrence

$$xp_n(x,t) = a_{n+1}(t)p_{n+1}(x,t) + b_n(t)p_n(x,t) + a_n(t)p_{n-1}(x,t). \tag{30}$$

The coefficients $a_n(t)$ and $b_n(t)$ satisfy the system of differential equations

$$\begin{aligned} \frac{d}{dt}a_n &= \tfrac{1}{2}a_n(b_n - b_{n-1}) \\ \frac{d}{dt}b_n &= a_{n+1}^2 - a_n^2. \end{aligned} \tag{31}$$

The system can be rewritten in the (more standard) equivalent form

$$\begin{aligned} \frac{d}{dt}a_n &= a_n(b_{n+1} - b_n) \\ \frac{d}{dt}b_n &= 2(a_n^2 - a_{n-1}^2). \end{aligned} \tag{32}$$

The paper [2] contains a nice introduction to this important `integrable system`.

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.

[2] G. Teschl. Almost everything you wanted to know about the Toda equation. *Jahresber. Deitsch. Math-Verein.*, 103:149–162, 2001.

[3] W. Van Assche. *Orthogonal polynomials and Painlevé equations*, volume 27 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, 2018.