THE BERNOULLI POLYNOMIALS

The **Bernoulli numbers** have been defined here by the exponential generating function

(1)
$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

The question considered here is how to introduce a *parameter* into these numbers. One option is to multiply the left-hand side of (1) by a simple function and to see what happens. Naturally, from the point of view of generating functions, the simplest possible function is an exponential. Therefore consider the expansion

(2)
$$\frac{t}{e^t - 1} \times e^{xt} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

where the coefficients on the right-hand side depend on this parameter x.

The rule to multiply the series on the left is

(3)
$$\left(\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \right) \times \left(\sum_{j=0}^{\infty} b_j \frac{t^j}{j!} \right) = \sum_{r=0}^{\infty} \left[\sum_{i=0}^{r} \binom{r}{i} a_i b_{r-i} \right] \frac{t^r}{r!}$$

and applied to (2) gives

(4)
$$\frac{t}{e^t - 1} \times e^{xt} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \times \left(\sum_{j=0}^{\infty} \frac{x^j t^j}{j!} \right)$$
$$= \sum_{r=0}^{\infty} \left[\sum_{i=0}^{r} \binom{r}{i} B_i x^{r-i} \right] \frac{t^r}{r!}.$$

It follows that

(5)
$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$$

is a polynomial in x, called the **Bernoulli polynomial**.

An alternative approach to these polynomials is to start with the generating function (2), without any assumptions on the functions $B_k(x)$ and compute the first one by

(6)
$$B_0(x) = \lim_{t \to 0} \frac{t e^{xt}}{e^t - 1} = 1$$

and then differentiate (2) with respect to x (after all the exponential function appears here, so this must be simple) to obtain

(7)
$$\sum_{k=0}^{\infty} B'_k(x) \frac{t^k}{k!} = \frac{t^2 e^{xt}}{e^t - 1}.$$

1

The right-hand side is

$$(8) \qquad t \times \frac{te^{xt}}{e^t - 1} \;=\; t \times \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

$$= \; \sum_{k=0}^{\infty} B_k(x) \frac{t^{k+1}}{k!}$$

$$= \; \sum_{k=1}^{\infty} B_{k-1}(x) \frac{t^k}{(k-1)!},$$

and using $B_0(x) = 1$ it is seen that the sum on the left-hand side starts at $k = 1$, leading to

$$(9) \qquad \sum_{k=1}^{\infty} B_k'(x) \frac{t^k}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x) \frac{t^k}{(k-1)!}.$$

Now match coefficients of equal powers of t to produce

$$(10) \qquad \frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!}$$

This is written as a theorem.

Theorem 1. *The functions $B_k(x)$ appearing in the expansion* (2) *satisfy the recurrence*

$$(11) \qquad B_k'(x) = k B_{k-1}(x), \quad \text{for } k \geq 1.$$

In particular, the initial condition $B_0(x) = 1$, shows that $B_k(x)$ is a polynomial of degree k. The expansion (2) *gives the special value*

$$(12) \qquad B_k(0) = B_k$$

that can be used as a condition to determine the constant of integration in the process of determining $B_k(x)$ from $B_{k-1}(x)$ using (11).

The presence of the exponential factor e^{xt} suggests to compute the generating functions of $B_k(x+1)$. This is

$$(13) \qquad \sum_{k=0}^{\infty} B_k(x+1) \frac{t^k}{k!} \;=\; \frac{te^{(x+1)t}}{e^t - 1}$$

$$= \; \frac{t(e^t - 1 + 1)e^{xt}}{e^t - 1}$$

$$= \; te^{xt} + \frac{te^{xt}}{e^t - 1}.$$

Now compare the coefficients of t^k to obtain the relation stated in the next theorem.

Theorem 2. *For $k \geq 0$, the Bernoulli polynomials satisfy the relation*

$$(14) \qquad B_k(x+1) = B_k(x) + kx^{k-1}.$$

Now that we see the expression (14) it seems natural to sum over x, after all some parts will telescope. To make it consistent with previous notation, change (14) to

$$(15) \qquad B_{a+1}(k+1) = B_{a+1}(k) + (a+1)k^a$$

(the pair $\{k, x\}$ was changed to $\{a+1, k\}$). Then, summing from $k = 0$ to n gives

$$(16) \qquad B_{a+1}(n+1) - B_{a+1}(0) = (a+1) \sum_{k=0}^{n} k^a.$$

that implies

$$(17) \qquad \sum_{k=1}^{n} k^a = \frac{B_{a+1}(n+1) - B_{a+1}}{a+1}.$$

This is the expression that we were looking for at the beginning of the semester.

Many other identities for the Bernoulli polynomials can be established using the generating function. The next example illustrates this point:

Theorem 3. *The Bernoulli polynomials satisfy the identity*

$$(18) \qquad B_k(x) + B_k\left(x + \tfrac{1}{2}\right) = 2^{1-k} B_k(2x), \qquad \text{for } k \geq 0.$$

The generating function of the left-hand side is

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\begin{aligned}
\sum_{k=0}^{\infty} \left[B_k(x) + B_k\left(x + \tfrac{1}{2}\right) \right] \frac{t^k}{k!} &= \frac{te^{xt}}{e^t - 1} + \frac{te^{(x+1/2)t}}{e^t - 1} \\
&= \frac{te^{xt}}{e^t - 1}\left(1 + e^{t/2}\right) \\
&= \frac{te^{xt}}{(e^{t/2} - 1)(e^{t/2} + 1)}\left(1 + e^{t/2}\right) \\
&= \frac{te^{xt}}{(e^{t/2} - 1)} \\
&= 2\frac{(t/2)e^{2x(t/2)}}{(e^{t/2} - 1)} \\
&= 2\sum_{k=0}^{\infty} B_k(2x)\frac{(t/2)^k}{k!} \\
&= \sum_{k=0}^{\infty} 2^{1-k} B_k(2x)\frac{t^k}{k!}
\end{aligned}
$$

and the result is obtained by matching powers of t.

The identity (5) shows that the value of the Bernoulli polynomial at $x = 0$ is the Bernoulli numbers, that is,

$$(19) \qquad B_k(0) = B_k.$$

Replacing this in (18) gives

$$(20) \qquad B_k + B_k\left(\tfrac{1}{2}\right) = 2^{1-k} B_k.$$

This proves the next identity.

Theorem 4. *The Bernoulli polynomials satisfy*

$$(21) \qquad B_k\left(\tfrac{1}{2}\right) = \left(2^{1-k} - 1\right) B_k, \qquad \text{for } k \geq 0.$$

With the value of $B_k\left(\frac{1}{2}\right)$ we can replace $x = \frac{1}{2}$ in (18) to obtain

$$(22) \qquad\qquad B_k\left(\tfrac{1}{2}\right) + B_k(1) = 2^{1-k}B_k(1).$$

This gives

$$(23) \qquad\qquad B_k\left(\tfrac{1}{2}\right) = \left(2^{1-k} - 1\right)B_k(1).$$

Comparing with (21) produces

$$(24) \qquad\qquad \left(2^{1-k} - 1\right)B_k(1) = \left(2^{1-k} - 1\right)B_k.$$

For $k \neq 1$ the common factor does not vanish, this gives $B_k(1) = B_k$. On the other hand

$$(25) \qquad\qquad B_k(1) = B_0 x + B_1 \qquad \text{at } x = 1$$

that produces

$$(26) \qquad\qquad B_k(1) = \tfrac{1}{2}.$$

This is summarized as follows

$$(27) \qquad\qquad B_k(1) = \begin{cases} B_k & \text{for } k \neq 1 \\ -B_k & \text{for } k = 1, \end{cases}$$

that can be written in the simpler form

$$(28) \qquad\qquad B_k(1) = (-1)^k B_k.$$

This is valid for $k = 1$ and k even. For k odd $k \neq 1$, the right-hand side of (28) vanishes and the same is true for the left-hand side from (27). Therefore (28) holds for all values of k.

Theorem 5. *For $k \geq 0$*

$$(29) \qquad\qquad B_k(1) = (-1)^k B_k.$$

It is easy to check that the relation (29) extends to the Bernoulli polynomials. Indeed, starting with the generating function

$$(30) \qquad\qquad \sum_{k=0}^{\infty} B_k(1-x)\frac{t^k}{k!} = \frac{te^{(1-x)t}}{e^t - 1}$$

and writing $s = -t$ it becomes

$$(31) \qquad\qquad \begin{aligned} \frac{te^{(1-x)t}}{e^t - 1} &= -\frac{se^{-s(1-x)}}{e^{-s} - 1} \\ &= \frac{se^{xs}}{e^s - 1} \\ &= \sum_{k=0}^{\infty} B_k(x)\frac{s^k}{k!} \\ &= \sum_{k=0}^{\infty} B_k(x)\frac{(-1)^k t^k}{k!}. \end{aligned}$$

Now compare the coefficients of t^k to obtain the next result.

Theorem 6. *The Bernoulli polynomials satisfy*

$$(32) \qquad\qquad B_k(1-x) = (-1)^k B_k(x), \qquad \text{for } k \geq 0.$$