

# THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

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ABSTRACT. We analyze properties of the 2-adic valuations of an integer sequence that originates from an explicit evaluation of a quartic integral.

## 1. INTRODUCTION

The sequence of positive integers

$$(1.1) \quad b_{l,m} = \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

for  $m \in \mathbb{N}$  and  $0 \leq l \leq m$  appears in the evaluation of the definite integral

$$(1.2) \quad N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.$$

In [1] it was shown that the polynomial defined by

$$P_m(a) := 2^{-2m} \sum_{l=0}^m b_{l,m} a^l$$

satisfies

$$(1.3) \quad P_m(a) = \frac{1}{\pi} 2^{m+3/2} (a+1)^{m+1/2} N_{0,4}(a; m).$$

The evaluation of  $b_{l,m}$  using (1.1) is efficient if  $l$  is close to  $m$ . For instance,

$$(1.4) \quad b_{m,m} = 2^m \binom{2m}{m} \quad \text{and} \quad b_{m-1,m} = 2^{m-1} (2m+1) \binom{2m}{m}.$$

An expression for a closely related integer sequence,  $A_{l,m}$ , was established in [2] and is given in the next theorem.

**Theorem 1.1.** *Define*

$$(1.5) \quad A_{l,m} := \frac{l! m!}{2^{m-l}} b_{l,m}.$$

*Then there exist polynomials  $\alpha_l(m)$  and  $\beta_l(m)$ , with positive integer coefficients, such that*

$$(1.6) \quad A_{l,m} = \alpha_l(m) \prod_{k=1}^m (4k-1) - \beta_l(m) \prod_{k=1}^m (4k+1).$$

*The degrees of  $\alpha_l$  and  $\beta_l$  are  $l$  and  $l-1$ , respectively.*

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This formula leads to an efficient method of evaluating the coefficients  $A_{l,m}$  if  $l$  is small. For instance,

$$(1.7) \quad A_{0,m} = \prod_{k=1}^m (4k-1),$$

$$(1.8) \quad A_{1,m} = (2m+1) \prod_{k=1}^m (4k-1) - \prod_{k=1}^m (4k+1).$$

The polynomial

$$(1.9) \quad X_l(s) := \alpha_l \left( \frac{s-1}{2} \right)$$

satisfies the recurrence

$$(1.10) \quad X_{l+1}(s) = 2sX_l(s) - (s^2 - (2l-1)^2)X_{l-1}(s),$$

with initial conditions  $X_0(s) = 1$ ,  $X_1(s) = s$ . Similarly,

$$(1.11) \quad Y_l(s) := \beta_l \left( \frac{s-1}{2} \right)$$

satisfies the same recurrence (1.10) but with initial conditions  $Y_0(s) = 0$ ,  $Y_1(s) = 1$ . This was used by John Little [9] to establish the next theorem.

**Theorem 1.2.** *All the zeros of  $\alpha_l(m)$  and  $\beta_l(m)$  lie on the line  $\operatorname{Re} m = -\frac{1}{2}$ .*

In this paper we study arithmetical properties of the sequence  $\{b_{l,m}\}$ , or equivalently,  $\{A_{l,m}\}$ . Henceforth we assume that the index  $l \in \mathbb{N}$  is fixed and  $m \geq l$ .

The results described in this paper started as empirical observations on the behavior of the highest power of 2 that divides the numbers  $A_{l,m}$ . For any integer  $x$ , we denote this power by  $\nu_2(x)$ . We now give an algorithm that will start with the sequence  $\{\nu_2(A_{l,m}) : m \geq l\}$  and produce at the end a constant sequence. Here we only illustrate this algorithm with the case  $l = 59$  and introduce some terminology. The full details are discussed in Section 4.

Figure 1 shows the graph of  $\nu_2(A_{59,m})$  for  $59 \leq m \leq 197$ . The horizontal axis is  $m - 58$ , so the indexing starts at 1.

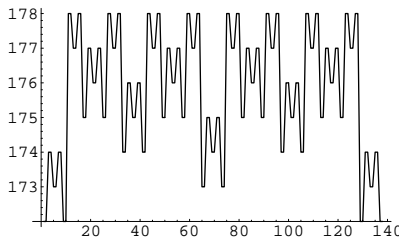


FIGURE 1. The 2-adic valuation of  $A_{59,m}$  for  $59 \leq m \leq 196$

The figure suggests that the values of  $\{\nu_2(A_{59,m}) : m \geq 59\}$  has a *block structure* meaning that it is composed of consecutive blocks, all of the same length. Indeed,  $\nu_2(A_{59,m})$  begins with

$$\{172, 172, 174, 174, 173, 173, 174, 174, 172, 172, 178, 178, 177, 177, \dots\}.$$

This motivates the next definition.

**Definition 1.3.** Let  $s \in \mathbb{N}$ ,  $s \geq 2$ . We say that a sequence  $\{a_j : j \in \mathbb{N}\}$  is *simple of length  $s$*  ( or  *$s$ -simple*) if, for each  $t \in \{0, 1, 2, \dots\}$ , we have

$$(1.12) \quad a_{st+1} = a_{st+2} = \dots = a_{s(t+1)}.$$

The sequence  $\{a_j : j \in \mathbb{N}\}$  is said to have a *block structure* if it is  $s$ -simple for some  $s \geq 2$ . The *jump* at  $j$  is defined by  $a_{j+1} - a_j$ .

**Note.** The main result, presented as Theorem 2.3, relates the 2-adic valuation of  $A_{l,m}$  to that of a Pochhammer symbol, namely

$$(1.13) \quad \nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l.$$

This permits us to simplify the difference of their consecutive values. Theorem 3.1 states

$$(1.14) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1).$$

We will use this fact to establish in Theorem 3.2 that the sequence  $\{\nu_2(A_{l,m}) : m \geq l\}$  is  $2^{1+\nu_2(l)}$ -simple.

We now proceed to the second step of the algorithm; given that  $\{\nu_2(A_{59,m}) : m \geq 59\}$  is 2-simple, we take every other term of this sequence to eliminate repetitions. The result appears in Figure 2 and the data begins with

$$\{172, 174, 173, 174, 172, 178, 177, \dots\}.$$

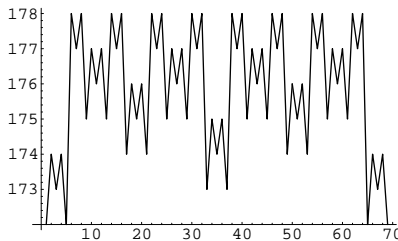


FIGURE 2. Every other term from Figure 1

The third step is to subtract from Figure 2 the 2-adic valuation of the index  $m$ . This produces data starting with

$$\{172, 173, 173, 172, 172, 177, 177, 175, 175, 176, 176, 175, 175, 177, 177, \dots\}.$$

This set almost has a block structure, except that the first element appears only once. We now add this extra element to produce a genuine block structure. The resulting sequence, shown in Figure 3, is 2-simple.

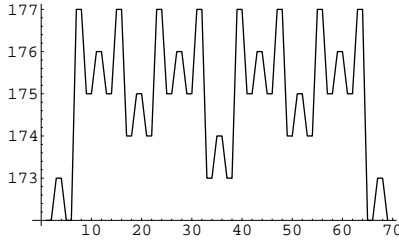
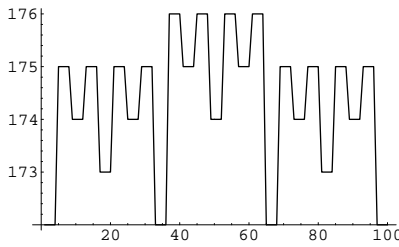
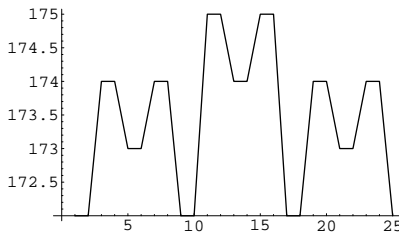


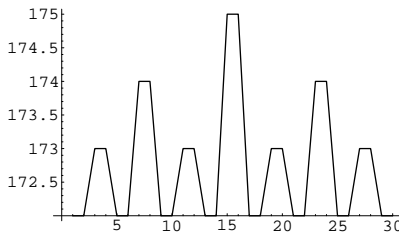
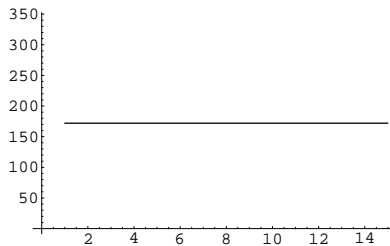
FIGURE 3. The end of the first cycle

We will show that at this point one always gets a block structure. *The length of the block is not necessarily the same in each cycle.* This is the end of the cycle. We now go back to the first step and begin a new cycle.

The main goal of this paper is to describe this algorithm and to prove that it yields a constant sequence after a finite number of steps. The next figures display the sequences at the end of each cycle of the algorithm for  $l = 59$ .

FIGURE 4. End of the second cycle for  $l = 59$ FIGURE 5. End of the third cycle for  $l = 59$ 

**The algorithm - in detail.**

FIGURE 6. End of the fourth cycle for  $l = 59$ FIGURE 7. End of the fifth cycle for  $l = 59$ 

**The maps  $F$  and  $T$ .** The algorithm requires two operators defined on sequences:

$$(1.15) \quad F(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_1, a_2, a_3, \dots\},$$

and

$$(1.16) \quad T(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_3, a_5, a_7, \dots\}.$$

Now introduce the sequence  $c$  by

$$(1.17) \quad c := \{\nu_2(m) : m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \dots\}.$$

**The algorithm:**

- 1) Start with the sequence  $X_1(l) := \{\nu_2(A_i(m)) : m \geq l\}$ .
- 2) Find  $n_1 \in \mathbb{N}$  so that the sequence  $X_1(l)$  is  $2^{n_1}$ -simple. Define  $Y_1(l) := T^{n_1}(X_1(l))$ . At the initial stage, Theorem 3.2 shows that  $n_1 = 1 + \nu_2(l)$ .
- 3) Introduce the shift  $Z_1(l) := Y_1(l) - c$ .
- 4) Define  $X_2(l) := F(Z_1(l))$ .

The sequence  $X_2$  is  $2^{n_2}$ -simple. Then return to step 2) with  $X_2$  instead of  $X_1$ .

**Definition 1.4.** Let  $\omega(l)$  be the number of steps required for the algorithm to yield a constant sequence. The sequence of integers

$$(1.18) \quad \Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\}$$

TABLE 1. Reduction sequence for  $1 \leq l \leq 15$ .

$l$	binary form	$\Omega(l)$
4	100	3
5	101	12
6	110	21
7	111	111
8	1000	4
9	1001	13
10	1010	22
11	1011	112
12	1100	31
13	1101	121
14	1110	211
15	1111	1111

is called the *reduction sequence* of  $l$ . The number  $\omega(l)$  will be called the *reduction length* of  $l$ . The constant sequence obtained after  $\omega(l)$  steps is called the *reduced constant*.

We prove in Corollary 4.4 that  $\omega(l) < \infty$ . Therefore the algorithm yields a constant sequence in a finite number of steps.

Table 1 shows the results of the algorithm for  $4 \leq l \leq 15$ .

We also provide a combinatorial interpretation of  $\Omega(l)$ . This requires the composition of the index  $l$ .

**Definition 1.5.** Let  $l \in \mathbb{N}$ . The *composition* of  $l$ , denoted by  $\Omega_1(l)$ , is defined as follows: write  $l$  in binary form. Read the sequence from right to left. The first part of  $\Omega_1(l)$  is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

For example,

$$(1.19) \quad \Omega_1(13) = \{1, 2, 1\}.$$

and

$$(1.20) \quad \Omega_1(14) = \{2, 1, 1\}.$$

Observe that  $\Omega_1(13) = \Omega(13)$  and  $\Omega_1(14) = \Omega(14)$ . We claim that this is always true.

**Theorem 1.6.** *The reduction sequence  $\Omega(l)$  associated to an integer  $l$  is the sequence of compositions of  $l$ , that is,*

$$(1.21) \quad \Omega(l) = \Omega_1(l)$$

This theorem is proved in Section 4.

2. THE 2-ADIC VALUATIONS OF  $A_{l,m}$ 

Given a prime  $p$  and a rational number  $r$ , there exist unique integers  $a, b, c$  with  $a$  and  $b$  not divisible by  $p$  and  $\gcd(a, b) = 1$  such that

$$(2.1) \quad r = \frac{a}{b} p^c.$$

The integer  $c$  is the  $p$ -adic valuation of  $r$  and we denote it by  $\nu_p(r)$ . Observe that we depart from the usual convention  $c = -\nu_p(r)$ . There are many well-known elementary results concerning the  $p$ -adic valuations of integers, such as

$$(2.2) \quad \nu_p(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{p^k} \right\rfloor.$$

Naturally this sum is finite, ending at  $k = \lfloor \log_2 m \rfloor$ . A famous result of Legendre [4, 7] states that

$$(2.3) \quad \nu_p(m!) = \frac{m - s_p(m)}{p - 1},$$

where  $s_p(m)$  is the sum of the base- $p$  digits of  $m$ . In particular,

$$(2.4) \quad \nu_2(m!) = m - s_2(m).$$

Kummer's result

$$(2.5) \quad \nu_2 \left( \binom{m}{k} \right) = s_2(k) + s_2(m - k) - s_2(m),$$

follows directly from here.

We describe divisibility properties of the sequence  $\{A_{l,m}\}$ . As the factorization of factorials is elementary, this provides similar properties of  $\{b_{l,m}\}$ .

**Lemma 2.1.** *For the prime  $p = 2$ , we have*

$$(2.6) \quad \nu_2(A_{l,m}) = \nu_2(l!) + \nu_2(m!) - m + l + \nu_2(b_{l,m}),$$

and for  $p$  odd,

$$(2.7) \quad \nu_p(A_{l,m}) = \nu_p(l!) + \nu_p(m!) + \nu_p(b_{l,m}).$$

The 2-adic value of  $b_{0,m}$  follows directly from (1.7) and (2.6). Clearly  $A_{0,m}$  is odd, so  $\nu_2(A_{0,m}) = 0$ . Therefore  $\nu_2(b_{0,m}) = m - \nu_2(m!)$  and Legendre's result (2.4) reduces this to

$$(2.8) \quad \nu_2(b_{0,m}) = s_2(m).$$

The coefficients  $b_{1,m}$  were analyzed in [2] using formula (1.8). The main result is:

**Theorem 2.2.** *The 2-adic valuation of  $A_{1,m}$  is given by*

$$(2.9) \quad \nu_2(A_{1,m}) = \nu_2(2m(m+1)) = \nu_2(m(m+1)) + 1.$$

Then (2.6) yields

$$(2.10) \quad \nu_2(b_{1,m}) = s_2(m) + \nu_2(m(m+1)).$$

The key elements of the proof are to expand the products in (1.8) using the Stirling numbers of the first kind, whose generating function is

$$(2.11) \quad x(x+1)(x+2)\cdots(x+r-1) = \sum_{k=0}^r \begin{bmatrix} r \\ k \end{bmatrix} x^k,$$

and to rewrite the resulting sums using the representation

$$(2.12) \quad \begin{bmatrix} r \\ r-k \end{bmatrix} = \sum_{i=0}^{k-1} \binom{r}{2k-i} C_{k,i}$$

for some integers  $C_{k,i}$ .

The first goal of this paper is to present the following generalization of Theorem 2.2.

**Theorem 2.3.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$(2.13) \quad \nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l,$$

where  $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$  is the Pochhammer symbol.

*Proof.* We present two different proofs.

**First proof.** Define the numbers

$$(2.14) \quad B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}.$$

We need to prove that  $B_{l,m}$  is odd. The WZ-method [10] shows that the numbers  $b_{l,m}$  satisfy the recurrence

$$(2.15) \quad b_{l+1,m} = \frac{2m+1}{l+1} b_{l,m} - \frac{(m+l)(m+1-l)}{l(l+1)} b_{l-1,m},$$

and the relation

$$(2.16) \quad B_{l,m} = \frac{l! m! (m-l)!}{2^m (m+l)!} b_{l,m}$$

implies that

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

The initial values  $B_{m,m} = 1$  and  $B_{m-1,m} = 2m+1$  show that  $B_{l,m}$  is an odd integer as required.

**Second proof.** We have

$$(2.17) \quad \nu_2(A_{l,m}) = l + \nu_2 \left( \sum_{k=l}^m T_{m,k} \frac{(m+k)!}{(m-k)!(k-l)!} \right),$$

where

$$(2.18) \quad T_{m,k} = \frac{(2m-2k)!}{2^{m-k} (m-k)!}.$$



The identity

$$(2.19) \quad T_{m,k} = \frac{(2(m-k))!}{2^{m-k}(m-k)!} = (2m-2k-1)(2m-2k-3)\cdots 3 \cdot 1$$

shows that  $T_{m,k}$  is an odd integer. Then (2.17) can be written as

$$\begin{aligned} \nu_2(A_{l,m}) &= l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m+k+l)!}{(m-k-l)!k!} \right) \\ &= l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!} \right). \end{aligned}$$

The term corresponding to  $k=0$  is singled out as we write

$$\nu_2(A_{l,m}) = l + \nu_2 \left( T_{m,l}(m-l+1)_{2l} + \sum_{k=1}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!} \right).$$

The claim

$$(2.20) \quad \nu_2 \left( \frac{(m-k-l+1)_{2k+2l}}{k!} \right) > \nu_2((m-l+1)_{2l})$$

will complete the proof.

To prove (2.20) we use the identity

$$\frac{(m-k-l+1)_{2k+2l}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_k (m+l+1)_k}{k!}$$

and the fact that the product of  $k$  consecutive numbers is always divisible by  $k!$ .

This follows from the identity

$$(2.21) \quad \frac{(a)_k}{k!} = \binom{a+k-1}{k}.$$

Now if  $m+l$  is odd,

$$(2.22) \quad \nu_2 \left( \frac{(m-l-k+1)_k}{k!} \right) \geq 0 \text{ and } \nu_2((m+l+1)_k) > 0,$$

and if  $m+l$  is even

$$(2.23) \quad \nu_2 \left( \frac{(m+l+1)_k}{k!} \right) \geq 0 \text{ and } \nu_2((m-l-k+1)_k) > 0.$$

This proves (2.20) and establishes the theorem.  $\square$

**Corollary 2.4.** *The 2-adic valuation of  $A_{l,m}$  is given by*

$$(2.24) \quad \nu_2(A_{l,m}) = 3l - s_2(m+l) + s_2(m-l).$$

*Proof.* This follows directly from (2.13) and Legendre's result (2.4).  $\square$

3. PROPERTIES OF THE FUNCTION  $\nu_2(A_{l,m})$ 

In this section we describe properties of the function  $\nu_2(A_{l,m})$  for  $l$  fixed and  $m \geq l$ . In particular, we show that each of these sequences has a block structure. The initial value

$$(3.1) \quad b_{l,l} = 2^l \binom{2l}{l}$$

yields

$$A_{l,l} = 2^l (2l)!,$$

so that

$$\begin{aligned} \nu_2(A_{l,l}) &= l + \nu_2((2l)!) = l + (2l - s_2(2l)) \\ &= 3l - s_2(l). \end{aligned}$$

This is consistent with Corollary 2.4.

The next element in the sequence is given by

$$(3.2) \quad A_{l,l+1} = \frac{1}{2} l! (l+1)! b_{l,l+1},$$

and (1.4) yields

$$(3.3) \quad A_{l,l+1} = (2l+3)(2l+1)A_{l,l}.$$

Therefore,

$$(3.4) \quad \nu_2(A_{l,l+1}) = \nu_2(A_{l,l}).$$

This is again consistent with Corollary 2.4 in view of

$$(3.5) \quad s_2(2l+1) = s_2(2l) + 1 = s_2(l) + 1.$$

We generalize (3.4) in the following theorem.

**Theorem 3.1.** *Let  $l \in \mathbb{N}$  be fixed. Then for  $m \geq l$ , we have*

$$(3.6) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m+l+1) - \nu_2(m-l+1).$$

*Proof.* The expression of Corollary 2.4 yields

$$(3.7) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \psi(m+l) - \psi(m-l),$$

where  $\psi(n) := s_2(n) - s_2(n+1)$ . We now show that  $\psi(n) = \nu_2(n+1) - 1$ , to complete the proof. To prove this, assume that the binary representation of  $n$  begins with  $N$  consecutive ones; that is,

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots + n_{N-1} 2^{N-1} + 0 \cdot 2^N + n_{N+1} 2^{N+1} + \cdots + n_r 2^r,$$

where  $n_0 = n_1 = \cdots = n_{N-1} = 1$ . Then

$$(3.8) \quad n+1 = 2^N + \sum_{j=N+1}^r n_j 2^j$$

is the binary representation of  $n+1$ . Therefore,  $N = \nu_2(n+1)$  and

$$s_2(n) = N + \sum_{j=N+1}^r n_j j, \quad s_2(n+1) = 1 + \sum_{j=N+1}^r n_j,$$

giving the result.

A second proof uses Legendre's formula (2.4). We have

$$\begin{aligned}
\nu_2(m) &= \nu_2(m!/(m-1)!) \\
&= \nu_2(m!) - \nu_2((m-1)!) \\
&= m - s_2(m) - (m-1 - s_2(m-1)) \\
&= s_2(m-1) - s_2(m) + 1,
\end{aligned}$$

as claimed.  $\square$

The fact that  $\nu_2(A_{l,l}) = \nu_2(A_{l,l+1})$  is a special case of Theorem 3.1:

$$(3.9) \quad \nu_2(A_{l,l+1}) - \nu_2(A_{l,l}) = \nu_2(2l+1) - \nu_2(1) = 0.$$

The value  $\nu_2(A_{l,l+2})$  can be obtained from

$$\begin{aligned}
\nu_2(A_{l,m+2}) - \nu_2(A_{l,m}) &= \nu_2(A_{l,m+2}) - \nu_2(A_{l,m+1}) \\
&\quad + \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) \\
&= \nu_2(m+l+2) - \nu_2(m-l+2) \\
&\quad + \nu_2(m+l+1) - \nu_2(m-l+1),
\end{aligned}$$

and with  $m = l$ , we get

$$\begin{aligned}
\nu_2(A_{l,l+2}) - \nu_2(A_{l,l}) &= \nu_2(2l+2) - \nu_2(2) + \nu_2(2l+1) - \nu_2(1) \\
&= \nu_2(l+1).
\end{aligned}$$

Thus the jump of the function  $\nu_2(A_{l,l+2})$  from the initial value  $\nu_2(A_{l,l})$  depends on the index  $l$ . If  $l$  is even, we have

$$\nu_2(A_{l,l+2}) = \nu_2(A_{l,l+1}) = \nu_2(A_{l,l}).$$

The reader will easily check that, in this case,  $\nu_2(A_{l,l+3})$  also has this value. Therefore, for  $l$  even, the function  $\nu_2(A_{l,m})$  begins with an interval of length 4 on which it is constant.

The first few values of  $\nu_2(A_{1,m})$  are given by

$$(3.10) \quad \nu_2(A_{1,m}) = \{2, 2, 3, 3, 2, 2, 4, 4, 2, 2, \dots\}.$$

We observe that this sequence is 2-simple. Formula (2.9) can be used to prove this property. Indeed, for  $m$  odd, we have

$$\nu_2(A_{1,m}) = \nu_2(2m(m+1)) = 1 + \nu_2(m+1)$$

and

$$\nu_2(A_{1,m+1}) = \nu_2(2(m+1)(m+2)) = 1 + \nu_2(m+1).$$

We have a similar block structure for  $l \geq 2$ .

**Theorem 3.2.** *The set  $\{\nu_2(A_{l,m}) : m \geq l\}$  is an  $s$ -simple sequence with  $s = 2^{1+\nu_2(l)}$ .*

*Proof.* For fixed  $k, l \in \mathbb{N}$ , define the sets of  $2^\mu$  integers

$$(3.11) \quad C_{k,l} = \{l + k \cdot 2^\mu + j : 0 \leq j \leq 2^\mu - 1\},$$

where  $\mu = 1 + \nu_2(l)$ . For example, if  $l$  is odd, then  $\mu = 1$  and

$$C_{k,l} = \{l + 2k + j : 0 \leq j \leq 1\} = \{l + 2k, l + 2k + 1\}.$$

It follows that

$$(3.12) \quad \{m \in \mathbb{N} : m \geq l\} = \bigcup_{k \geq 0} C_{k,l};$$

that is, the  $C_{k,l}$  partition the integers  $m \geq l$ . We claim that the function  $\nu_2(A_{l,m})$  is constant on each  $C_{k,l}$  and that the constant differs from  $C_{k,l}$  to  $C_{k+1,l}$ . We start with the case  $\nu_2(l) = 0$ . Then  $C_{k,l} = \{l + 2k, l + 2k + 1\}$ , and we use (3.6) to show that

$$\begin{aligned} \nu_2(A_{l,l+2k+1}) - \nu_2(A_{l,l+2k}) &= \nu_2(l + 2k + l + 1) - \nu_2(l + 2k - l + 1) \\ &= \nu_2(2l + 2k + 1) - \nu_2(2k + 1) = 0, \end{aligned}$$

and

$$\begin{aligned} \nu_2(A_{l,l+2k+2}) - \nu_2(A_{l,l+2k+1}) &= \nu_2(2l + 2k + 2) - \nu_2(2k + 2) \\ &= \nu_2(l + k + 1) - \nu_2(k + 1) \neq 0 \end{aligned}$$

because the numbers  $l + k + 1$  and  $k + 1$  have different parity.

The case  $\nu_2(l) = 1$  illustrates the general argument. Now the sets are

$$(3.13) \quad C_{k,l} = \{l + 4k, l + 4k + 1, l + 4k + 2, l + 4k + 3\},$$

and the jumps are evaluated as

$$(3.14) \quad \text{jump}_j := \nu_2(2l + 4k + j + 1) - \nu_2(4k + j + 1)$$

for  $0 \leq j \leq 2$ . In the case  $j$  even, then clearly  $\text{jump}_j = 0$ . If  $j$  is odd, namely  $j = 1$ , we write  $j = 2j_1 + 1$  and

$$\begin{aligned} \text{jump}_j &= \nu_2(2l + 4k + 2j_1 + 2) - \nu_2(4k + 2j_1 + 2) \\ &= \nu_2(l + 2k + j_1 + 1) - \nu_2(2k + j_1 + 1) \\ &= \nu_2(l + 2k + 1) - \nu_2(2k + 1), \end{aligned}$$

and this vanishes again. The jump between different the last element of  $C_{k,l}$  and the first one of  $C_{k+1,l}$  is given by the value  $j = 3$  in (3.14); we have

$$(3.15) \quad \nu_2(2l + 4k + 4) - \nu_2(4k + 4) = \nu_2(l/2 + k + 1) - \nu_2(k + 1) \neq 0$$

since the numbers  $l/2 + k + 1$  and  $k + 1$  differ by the odd value  $l/2$ .

We now consider the general case. The jump is given by

$$\text{jump}_j = \nu_2(2l + k \cdot 2^\mu + j + 1) - \nu_2(k \cdot 2^\mu + j + 1),$$

where  $0 \leq j \leq 2^\mu - 2$ . We need to check that  $\text{jump}_j = 0$  for  $0 \leq j \leq 2^\mu - 2$  and  $\text{jump}_j \neq 0$  for  $j = 2^\mu - 1$ . The vanishing of  $\text{jump}_j$  is clear if  $j$  is even because  $\mu = 1 + \nu_2(l) \geq 1$ . In the case  $j$  odd, we write  $j = 2j_1 + 1$  with  $0 \leq j_1 \leq 2^{\mu-1} - 2$ . Then

$$\text{jump}_j = \nu_2(l + k \cdot 2^{\mu-1} + j_1 + 1) - \nu_2(k \cdot 2^{\mu-1} + j_1 + 1).$$

This process can be repeated to obtain

$$\text{jump}_j = \nu_2(l2^{-\alpha} + k \cdot 2^{\mu-1-\alpha} + j_{1+\alpha} + 1) - \nu_2(k \cdot 2^{\mu-1-\alpha} + j_{1+\alpha} + 1),$$

for  $0 \leq j_{1+\alpha} \leq 2^{\mu-1-\alpha} - 2$ . The final step corresponds to  $\alpha = \mu - 2 = \nu_2(l) - 1$ . Then the range of  $j$  reduces to the single value  $j = 0$  and we have that

$$\text{jump}_j = \nu_2(2l2^{-\nu_2(l)} + 2k + 1) - \nu_2(2k + 1),$$

which vanishes because both terms on the right hand side are 2-adic valuations of odd integers.

In order to check that  $\text{jump}_{2^\mu-1} \neq 0$ , we write  $l = 2^{\mu-1}L$ , with  $L$  odd. Then

$$\begin{aligned} \text{jump}_{2^\mu-1} &= \nu_2(2l + 2^\mu k + 2^\mu) - \nu_2(2^\mu k + 2^\mu) \\ &= \nu_2(2^\mu L + 2^\mu k + 2^\mu) - \nu_2(2^\mu k + 2^\mu) \\ &= \nu_2(L + k + 1) - \nu_2(k + 1) \neq 0, \end{aligned}$$

because  $L + k + 1$  and  $k + 1$  differ by an odd integer.  $\square$

#### 4. THE ALGORITHM AND ITS COMBINATORIAL INTERPRETATION

The graphs of the function  $\nu_2(A_{l,m})$ , where we take every other  $2^{1+\nu_2(l)}$ -element to reduce the repeating blocks to a single value, are shown in the next figures. The main experimental result is that these graphs have an *initial segment* from which the rest is determined by adding a *central piece* followed by a *folding rule*. For example, in the case  $l = 1$ , the first few values of the reduced table are

$$\{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, \dots\}.$$

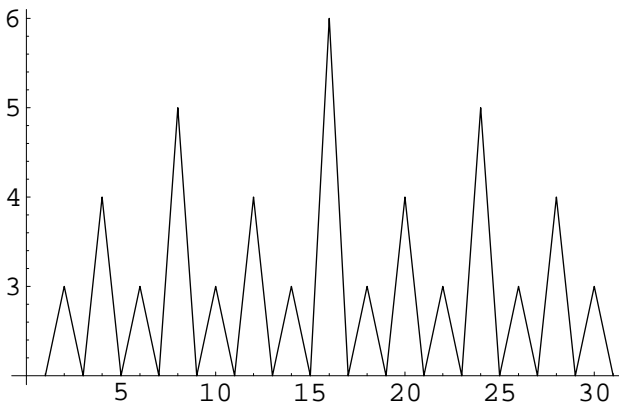


FIGURE 8. The 2-adic valuation of  $A_{1,m}$

The ingredients are:

*initial segment*:  $\{2, 3, 2\}$ ,

*central piece*: the value at the center of the initial segment, namely 3.

*rules of formation:* start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

$$\begin{aligned} \{2, 3, 2\} &\rightarrow \{2, 3, 2, 4\} \rightarrow \{2, 3, 2, 4, 2, 3, 2\} \rightarrow \{2, 3, 2, 4, 2, 3, 2, 5\} \rightarrow \\ &\rightarrow \{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, 2, 4, 2, 3, 2\}. \end{aligned}$$

The details are shown in Figure 8.

At the moment, there is no way to predict the initial segment nor the central piece. Figure 9 shows the beginning of the case  $l = 9$ . From here one could be tempted to predict that this graph extends as in the case  $l = 1$ . This is not correct, as can be seen in Figure 10. The new pattern described seems to be the correct one, as shown in Figure 11.

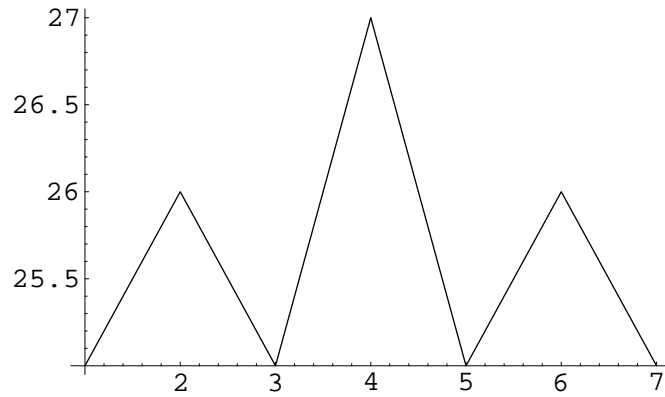


FIGURE 9. The beginning for  $l = 9$

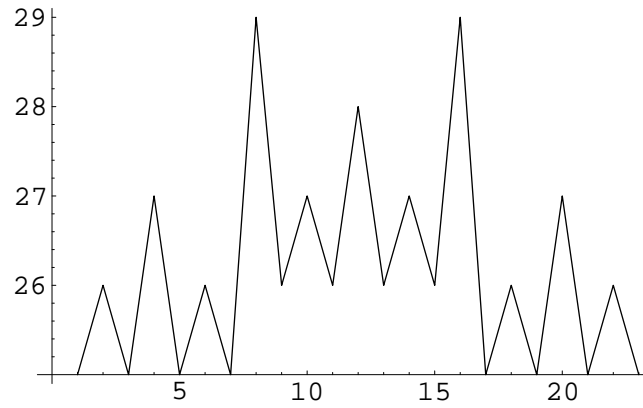
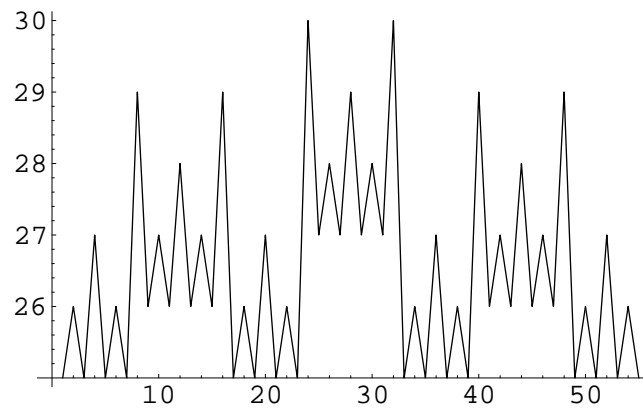
The initial pattern can be quite elaborate. Figure 12 illustrates the case  $l = 53$ . The proof of the Theorem 1.6 requires some preliminaries.

A) Given the values of  $\Omega_1(l)$  for  $2^j \leq l \leq 2^{j+1} - 1$ , the list for  $2^{j+1} \leq l \leq 2^{j+2} - 1$  is formed according to the following rule:

$l$  is even: add 1 to the first part of  $\Omega_1(l/2)$  to obtain  $\Omega_1(l)$ ;

$l$  is odd: prepend a 1 to  $\Omega_1(\frac{l-1}{2})$  to obtain  $\Omega_1(l)$ .

This is clear: if  $x_1x_2 \cdots x_t$  is the binary representation of  $l$ , then  $x_1x_2 \cdots x_t0$  is the one for  $2l$ . Thus, the first part of  $\Omega_1(2l)$  is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of  $\Omega_1(2l + 1)$ . The extra 1 that is placed at the end of the

FIGURE 10. The continuation of  $l = 9$ FIGURE 11. The pattern for  $l = 9$  persists

binary representation gives the first 1 in  $\Omega_1(2l + 1)$ .

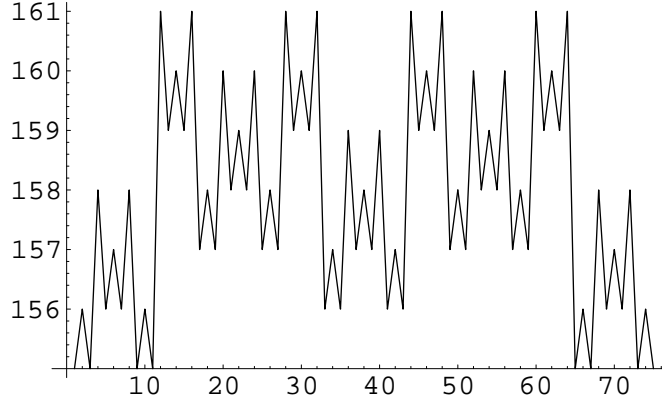
B) We now relate the 2-adic valuation of  $A_{2l,m}$  and  $A_{2l+1,m}$  to that of  $A_{l,m}$ .

**Proposition 4.1.** *Let  $l \in \mathbb{N}$ . Then*

$$(4.1) \quad \nu_2(A_{2l,m}) = 3l + \nu_2\left(A_{l, \lfloor \frac{m}{2} \rfloor}\right),$$

$$(4.2) \quad \nu_2(A_{2l+1,m}) = 3l + 2 + \nu_2(N - l) + \nu_2(A_{l,N}),$$

with  $N = \lfloor (m + 1)/2 \rfloor$ .

FIGURE 12. The initial pattern for  $l = 53$ 

*Proof.* Theorem 2.3 gives

$$\begin{aligned}
 \nu_2(A_{2l,2m}) &= \nu_2((2m - 2l + 1)_{4l}) + 2l \\
 &= \nu_2((2m - 2l + 1)(2m - 2l + 2) \cdots (2m + 2l - 1)(2m + 2l)) + 2l \\
 &= \nu_2(2^{2l}(m - l + 1)(m - l + 2) \cdots (m + l)) + 2l \\
 &= 4l + \nu_2((m - l + 1)_{2l}) \\
 &= 3l + \nu_2(A_{l,m}).
 \end{aligned}$$

A similar argument deals with  $\nu_2(A_{2l,2m+1})$  and (4.2).  $\square$

This proposition can be restated as follows.

**Proposition 4.2.** *Let*

$$(4.3) \quad \lambda_l = \frac{1 - (-1)^l}{2}, \quad M_0 = \lfloor \frac{m + \lambda_l}{2} \rfloor.$$

*Then*

$$(4.4) \quad \nu_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l \nu_2(M_0 - \lfloor l/2 \rfloor) + \nu_2(A_{\lfloor l/2 \rfloor, M_0}).$$

**Corollary 4.3.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$(4.5) \quad \nu_2(A_{l,m}) = 3l - s_2(l) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} \nu_2(M_k - \lfloor l/2^{k+1} \rfloor);$$

*where*

$$(4.6) \quad M_k = \lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k \lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \rfloor = \lfloor \frac{m + \sum_{n=0}^k 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \rfloor.$$

*Proof.* This is a repeated application of Proposition 4.2. The first term results from

$$\begin{aligned}
 \sum_{k \geq 0} \left( 2 \lfloor \frac{l}{2^k} \rfloor - \lfloor \frac{l}{2^{k+1}} \rfloor \right) &= 2l + \sum_{k \geq 1} \lfloor \frac{l}{2^k} \rfloor \\
 &= 2l + \nu_2(l!) \\
 &= 3l - s_2(l),
 \end{aligned}$$



in view of (2.4).  $\square$

**Corollary 4.4.** *The reduced constant is  $3l - s_2(l) = \nu_2(A_{l,l})$ .*

*Proof.* In the previous corollary, subtract the last term as per the reduction algorithm.  $\square$

We now restate Theorem 1.6.

**Theorem 4.5.** *Let  $\{k_1, \dots, k_n : 0 \leq k_1 < k_2 < \dots < k_n\}$ , be the unique collection of distinct positive integers such that*

$$(4.7) \quad l = \sum_{i=1}^n 2^{k_i}.$$

*Then the reduction sequence of  $l$  is  $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$ .*

*Proof.* The argument of the proof is to check that the rules of formation for  $\Omega_1(l)$  also hold for the reduction sequence  $\Omega(l)$ .

For  $l = 1$ , the block length is 2. This follows from Theorem 2.2, which states that

$$\nu_2(A_{1,2m-1}) = \nu_2(A_{1,2m}) = 1 + \nu_2(m).$$

After we extract every other term and subtract  $\{\nu_2(m) : m \geq 1\}$ , we obtain a constant sequence. Thus the algorithm terminates and the reduction sequence for  $l = 1$  is  $\Omega(1) = \{1\}$ .

The identity (4.1) in Proposition 4.1 shows that the sequence  $\{\nu_2(A_{2l,m}) : m \geq 2l\}$  is obtained from  $\{\nu_2(A_{l,m}) : m \geq l\}$  by doubling the block length and adding the constant  $3l$  to each element of the sequence. The addition of this constant does not affect the reduction sequence  $\Omega(l)$ , but the doubling of block length increases the first term of  $\Omega(l)$  by 1. Therefore

$$(4.8) \quad \Omega(2l) = \{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely what happens to the binary digits of  $l$ : if

$$l = \sum_{i=1}^n 2^{k_i}, \text{ then } 2l = \sum_{i=1}^n 2^{k_i+1}.$$

This concludes the argument for even indices.

Since  $2l + 1$  is odd, then the first term of the reduction sequence  $\Omega(2l + 1)$  is 1. The identity (4.2) in Proposition 4.1 states that

$$\nu_2(A_{2l+1,m}) = 3l + 1 + \nu_2(N - l) + \nu_2(A_{l,N}),$$

with  $N = \lfloor (m + 1)/2 \rfloor$ . After we extract the relevant subsequence, we obtain

$$(4.9) \quad \{3l + 1 + \nu_2(n) + \nu_2(A_{l,l+n}) : n \geq 1\},$$

and subtracting the dyadic valuation of the integers leaves

$$(4.10) \quad \{3l + 1 + \nu_2(A_{l,m}) : m \geq l + 1\}.$$

The first element of this sequence is  $\nu_2(A_{l,l+1})$ . This is also  $\nu_2(A_{l,l})$ , as shown in (3.4). Therefore, after the first step of the algorithm we have

$$(4.11) \quad \{3l + 1 + \nu_2(A_{l,m}) : m \geq l\};$$

this is a constant translate of the sequence  $\{\nu_2(A_{l,m}) : m \geq l\}$ . We conclude that, if the reduction sequence of  $l$  is

$$(4.12) \quad \{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\},$$

then that of  $2l + 1$  is

$$(4.13) \quad \{1, k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}.$$

This is precisely the behavior of  $\Omega_1$ . The proof is complete.  $\square$

**Corollary 4.6.** *The set  $\Omega(l)$  has cardinality*

$$(4.14) \quad s_2(l) = \text{the number of ones in the binary expansion of } l.$$

**Note.** The function  $s_2(l)$  has recently appeared in a different divisibility problem. In these papers it is denoted by  $d(l)$ . Lengyel [8] conjectured, and De Wannemacker [11] proved, that the 2-adic valuation of the Stirling numbers of the second kind  $S(n, k)$  is given by

$$(4.15) \quad \nu_2(S(2^n, k)) = s_2(k) - 1.$$

The Stirling numbers are given by the identity

$$(4.16) \quad x^n = \sum_{k=0}^n S(n, k)x(x-1)(x-2)\cdots(x-k+1)$$

and they count the number of ways to partition a set with  $n$  elements into exactly  $k$  nonempty subsets. De Wannemacker [12] also established the inequality

$$(4.17) \quad \nu_2(S(n, k)) \geq s_2(k) - s_2(n), \quad 0 \leq k \leq n.$$

The study of the 2-adic valuation of Stirling numbers suggests that

$$(4.18) \quad \nu_2(S(2^n + 1, k + 1)) = s_2(k) - 1,$$

which is a companion of (4.15).

**Remarks:**

Write  $l$  in the binary form:  $l = \sum_{j=1}^n 2^{k_j}$  with  $0 \leq k_1 < \dots < k_n$ . Then, for the  $M_k$  defined in (4.6) can be rewritten as

$$M_{k_i} = \lfloor \frac{m + \sum_{j=1}^i 2^{k_j}}{2^{1+k_i}} \rfloor.$$

1. In light of this, Corollary 4.3 may be given in the form

$$(4.19) \quad \nu_2(A_{l,m}) = 3l - s_2(l) + \sum_{i \geq 1} \nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor).$$

2.  $\nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor)$  is a  $2^{1+k_i}$ -simple sequence, i.e. of period  $2^{1+k_i}$ .

3.  $\nu_2(A_{l,m})$  inherits its  $2^{1+k_1}$ -simple structure from  $\nu_2(M_{k_1} - \lfloor l/2^{1+k_1} \rfloor)$ ; the term with the lowest period (or highest frequency) in the decomposition (4.19). Notice that this is consistent with Theorem 3.2, since  $k_1 = \nu_2(l)$ .

4. The sequence  $(\dots, \lambda_{\lfloor l/2^j \rfloor}, \lambda_l)$  is the binary code for  $l$ , and  $(\dots, k_2 + 1, k_1 + 1)$  are the exponents of 2 in the binary format for  $2l$ .

5. If we set  $k_0 := 0$ , we could reconstruct the sequence  $\nu_2(A_{l,m})$  by reverse engineering. Write the binary code for  $2l = \sum_{j=1}^n 2^{1+k_j}$ , and then, starting with the  $\infty$ -simple (constant) sequence  $3l - s_2(l)$ , then add the  $2^{1+k_1}, 2^{1+k_2}, \dots, 2^{1+k_n}$ -simple sequences  $\nu_2(M_{k_i} - \lfloor l/2^{1+k_i} \rfloor)$ . Here, the successive differences  $(1+k_j) - (1+k_{j-1}) = k_j - k_{j-1}$ , for  $j = 1, \dots, n$ , encode the *period switching-gaps* (or *indices of sequence shifting* as compared to the preceding stages) on the one hand, and the *integer composition* of  $2l$  on the other. This clearly confirms the bijective relationship between  $\Omega(l)$  and  $\Omega_1(l)$  that has been proven in Theorem 1.6.

## 5. GENERATING FUNCTIONS

In this section we list generating functions to describe existing interconnections between  $s_2(m)$  and the 2-adic valuation of  $A_{l,m}$  and also some miscellaneous concepts.

**Lemma 5.1.** *The generating function of  $\nu_2(m)$  is*

$$(5.1) \quad \sum_{m=1}^{\infty} \nu_2(m)x^m = \sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^k}}.$$

*Proof.* The right hand side is

$$\sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^k}} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x^{j \cdot 2^k}.$$

Now let  $m = j2^k$  and observe that the sum in  $k$  runs over all the  $\nu_2(m)$  powers of 2, except 1, that divide  $m$ .  $\square$

**Corollary 5.2.** *The generating function of  $\nu_2(m!)$  is*

$$(5.2) \quad \sum_{m=0}^{\infty} \nu_2(m!)x^m = \frac{1}{1-x} \sum_{k=1}^{\infty} \frac{x^{2^k}}{1-x^{2^k}}.$$

*Proof.* This follows from  $\nu_2(m) = \nu_2(m!) - \nu_2((m-1)!)$  and Lemma 5.1.  $\square$

**Note.** The generating function of the numbers

$$(5.3) \quad a_m := \nu_2(A_{1,m}) - 1 = \nu_2(m(m+1)),$$

given in Theorem 2.2, can be expressed as

$$(5.4) \quad \sum_{m \geq 1} a_m x^m = (1+x) \sum_{k \geq 1} \frac{x^{2^k-1}}{1-x^{2^k}}.$$

We have observed that the numbers  $a_m$  also appear in the well-known *Collatz* or  $3x + 1$  problem. Define a sequence by  $x_0(m) = m$  and let  $x_{k+1}(m) = T(x_k(m))$ , where

$$(5.5) \quad T(i) = \begin{cases} \frac{1}{2}i & \text{if } i \text{ is even,} \\ \frac{1}{2}(3i + 1) & \text{if } i \text{ is odd.} \end{cases}$$

The *orbit* of  $m \in \mathbb{N}$  is the set

$$(5.6) \quad \mathfrak{O}(m) := \{m, T(m), T^2(m), \dots\}.$$

The main conjecture for this problem is that *every orbit ends in the cycle*  $1 \rightarrow 2 \rightarrow 1$ . The reader will find in [5] an introduction to this problem and [3, 6] contain annotated bibliographies.

The connection with our work is given in the next theorem.

**Theorem 5.3.** *Let  $m \in \mathbb{N}$ . Then  $a_m := \nu_1(A_{1,m}) - 1 = \nu_2(m(m+1))$  is the first time at which the orbit  $\mathfrak{O}(m)$  changes parity. That is,*

$$(5.7) \quad m \equiv T(m) \equiv T^2(m) \equiv \dots \equiv T^{a_m-1}(m) \not\equiv T^{a_m}(m) \pmod{2}.$$

*Proof.* Suppose  $m$  is odd and write it as  $m = 2^j n - 1$ , with  $n$  odd. Then

$$(5.8) \quad j = \nu_2(m+1) \text{ and } n = \frac{m+1}{2^j}$$

are uniquely defined. Observe that

$$T(m) = T(2^j n - 1) = 3 \cdot 2^{j-1} n - 1$$

and for  $i < j$ ,

$$T^i(m) = T^i(2^j n - 1) = 3^i \cdot 2^{j-i} n - 1.$$

Finally,

$$T^j(m) = T^j(2^j n - 1) = 3^j n - 1.$$

To complete the proof, observe that

$$(5.9) \quad j = \nu_2(m+1) = \nu_2(m(m+1)) = N.$$

In the case  $m$  is even, write  $m = 2^t m_0$ , with  $m_0$  odd. Then

$$(5.10) \quad T^i(m) = 2^{t-i} m_0, \text{ for } 0 \leq i < t$$

and

$$(5.11) \quad T^t(m) = m_0.$$

The proof is completed by noticing that

$$(5.12) \quad t = \nu_2(m) = \nu_2(m(m+1)) = N.$$

□

For example take  $m = 63$ . Then  $x_1(63) = 95$ ,  $x_2(63) = 143$ ,  $x_3(63) = 215$ ,  $x_4(63) = 323$ ,  $x_5(63) = 485$ , and  $x_6(63) = 728$ . Thus,

$$(5.13) \quad \mathfrak{O}(63) = \{63, 95, 143, 215, 323, 485, \mathbf{728}, \dots\}.$$

It takes 6 iterations to produce an even entry. Observe that  $a_{63} = \nu_2((63)_2) = 6$ .

Similarly, we have

**Proposition 5.4.** *The first time the orbit of  $3^m - 1$  changes parity is after*

$$\nu_2(3^m(3^m - 1)) = \nu_2(3^m - 1) = \lambda_m + \nu_2(2m) = \nu_2(2^{1+\lambda_m}m)$$

*iterations.*

*Proof.* Use the binomial theorem for  $(2 + 1)^m - 1$ , while the generating function can be given by

$$(5.14) \quad \sum_{m \geq 1} \nu_2(3^m - 1)x^m = \frac{x^2}{1-x^2} + \sum_{k \geq 0} \frac{x^{2^k}}{1-x^{2^k}}.$$

□

**Lemma 5.5.** *The generating function of  $s_2(m)$  is*

$$(5.15) \quad \sum_{m=0}^{\infty} s_2(m)x^m = \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{x^{2^k}}{1+x^{2^k}}.$$

*Proof.* Legendre's identity (2.4) yields

$$(5.16) \quad s_2(m) - s_2(m-1) = 1 - \nu_2(m).$$

It follows that

$$(5.17) \quad \sum_{m=1}^{\infty} s_2(m) - \sum_{m=1}^{\infty} s_2(m-1)x^m = \sum_{m=1}^{\infty} x^m - \sum_{m=1}^{\infty} \nu_2(m)x^m.$$

The identity

$$(5.18) \quad \sum_{m=0}^{\infty} \frac{x^{2^m}}{1-x^{2^{m+1}}} = \frac{x}{1-x}$$

is equivalent to the fact that every positive integer  $n$  is of the form  $k \cdot 2^i$ , with  $k$  odd. Using this in (5.17) produces (5.15). □

**Lemma 5.6.** *Let  $N_l = 1 + \lfloor \log_2 l \rfloor$ . Then*

$$\begin{aligned} \sum_{m=l}^{\infty} (s_2(m-l) - s_2(m+l))x^{m-l} &= \frac{x^{2l} - 1}{1-x} \sum_{k=N_l}^{\infty} \frac{x^{2^k-2l}}{1+x^{2^k}} \\ &= \frac{2 \sinh(l \ln x)}{1-x} \sum_{k=N_l}^{\infty} \frac{x^{2^k-2l}}{1+x^{2^k}}. \end{aligned}$$

*Proof.* Simple application of Lemma 5.5. □

From this we obtain

**Theorem 5.7.** *The generating function of the sequence  $\nu_2(A_{l,m})$  is given by*

$$\begin{aligned} \sum_{m=l}^{\infty} \nu_2(A_{l,m})x^{m-l} &= \frac{3l}{1-x} + \frac{x^{2l} - 1}{1-x} \sum_{k=N_l}^{\infty} \frac{x^{2^k-2l}}{1+x^{2^k}} \\ &= \frac{1}{1-x} \left[ 3l + 2 \sinh(l \ln x) \sum_{k=N_l}^{\infty} \frac{x^{2^k-2l}}{1+x^{2^k}} \right]. \end{aligned}$$

*Proof.* The expression given here follows by elementary manipulations on the identity

$$(5.19) \quad \nu_2(A_{l,m}) = 3l - s_2(m+l) + s_2(m-l)$$

given in Corollary 2.4. □

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