

# A Simple Example of a New Class of Landen Transformations

Dante Manna and Victor H. Moll

**1. INTRODUCTION.** The method of completing squares yields an elementary procedure to evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}. \quad (1)$$

Write

$$ax^2 + bx + c = a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right],$$

and use a linear change of variables to obtain

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{2\pi}{\sqrt{4ac - b^2}}. \quad (2)$$

Observe that  $4ac - b^2 > 0$  is required for the convergence of (1).

The goal of this paper is to present a new proof of (2). We illustrate a technique that will apply to *any rational integrand*. Providing new proofs of an elementary result, such as (2), is usually an effective tool to introduce students to more interesting Mathematics. The method discussed here has a rich history that we describe in section 2.

It is an unfortunate fact that, despite our best efforts, evaluating definite integrals is not very much in fashion today. Thus we rephrase the previous evaluation as a question in dynamical systems: replace the parameters  $a$ ,  $b$ , and  $c$  in (1) with new ones given by the rules

$$\begin{aligned} a_{n+1} &= a_n \left[ \frac{(a_n + 3c_n)^2 - 3b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right], \\ b_{n+1} &= b_n \left[ \frac{3(a_n - c_n)^2 - b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right], \\ c_{n+1} &= c_n \left[ \frac{(3a_n + c_n)^2 - 3b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right], \end{aligned} \quad (3)$$

with  $a_0 = a$ ,  $b_0 = b$ , and  $c_0 = c$ . The reader is asked to check that (1) is invariant under (3), that is,

$$\int_{-\infty}^{\infty} \frac{dx}{a_{n+1}x^2 + b_{n+1}x + c_{n+1}} = \int_{-\infty}^{\infty} \frac{dx}{a_nx^2 + b_nx + c_n}, \quad (4)$$

and to prove that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = \frac{1}{2} \sqrt{4ac - b^2}, \quad \lim_{n \rightarrow \infty} b_n = 0. \quad (5)$$

Once this is done, we can pass to the limit in (4) and use the invariance of  $I$  to obtain

$$I = \frac{\pi}{\lim_{n \rightarrow \infty} a_n} = \frac{2\pi}{\sqrt{4ac - b^2}}. \quad (6)$$

This leads directly to a proof of (2). The advantage of this method is that it generalizes to integrands of higher degree.

We call (3) a *rational Landen transformation*. In section 2 we discuss the historical precedent and motivation behind such transformations. This history connects (6) to the magic of the arithmetic-geometric mean,  $\pi$ , and a wonderful numerical calculation of Gauss.

The rest of the paper is devoted to a detailed proof of the Landen transformation: the invariance of the rational integral (4) and the evaluation of the limits in (5). A scaling of the integrand that is a crucial step in producing this transformation is presented in section 3. The following section presents the trigonometrical aspects of this problem and completes the proof of (4). An algebraic calculation shows that the discriminant of the quadratic in (1) is preserved; that is,

$$4ac - b^2 = 4a_1c_1 - b_1^2. \quad (7)$$

This invariance is used in section 5 to analyze the dynamics of (3) and to establish (5).

**2. LANDEN TRANSFORMATIONS.** Many of the evaluations encountered in integral calculus illustrate the fact that definite integrals correspond to special values of functions. For example, the last integral in (2) is given by  $\pi = \tan^{-1}(\infty) - \tan^{-1}(-\infty)$ . Other special values appear in elementary courses:

$$\int_0^1 \frac{dx}{\sqrt{3-x^2}} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right). \quad (8)$$

The same is true for more complicated integrals. For instance, when  $0 < b < a < 1$ ,

$$G(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{1}{a} K(k), \quad (9)$$

with  $k^2 = 1 - b^2/a^2$ . Here  $K$  is the *complete elliptic integral of the first kind* defined by

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \quad (10)$$

Elliptic integrals appear at the center of classical analysis. Their name comes from the fact that they provide explicit formulas for the length of an ellipse.

The inverse of

$$f(z) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

is similar to  $\sin z$ , so (8) and (10) are not so different after all. This new function is the *elliptic sine* (or *sinus amplitudinus*) of Jacobi [15], denoted by  $\operatorname{sn} z$ . It completes the trilogy:  $\sin z$  (circular),  $\sinh z$  (hyperbolic), and  $\operatorname{sn} z$  (elliptic). The question of evaluating definite integrals sometimes comes down to how many functions one knows.

Our complaint that students today are exposed only to the most basic of functions is not new. Klein states [16, p. 294]: *When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.*<sup>1</sup>

Suppose that  $a$  and  $b$  are positive real numbers. It is not hard to check that the sequences  $\{a_n\}$  and  $\{b_n\}$  defined recursively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad (11)$$

$a_0 = a$ , and  $b_0 = b$  converge to a common limit: namely, the *arithmetic-geometric mean* of  $a$  and  $b$ , denoted by  $\operatorname{AGM}(a, b)$ . This is a fascinating function; the book [5] explains its connections with modern algorithms for the evaluation of  $\pi$ . The reader will find in [1] a survey of maps similar to (11) and an extensive bibliography.

At the turn of the eighteenth century, Gauss [13] was interested in lemniscates and their lengths. After a numerical calculation, he observed that

$$\frac{1}{\operatorname{AGM}(1, \sqrt{2})}$$

and

$$\frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

agree to eleven decimal places. (The integral gives the length of a lemniscate.) With remarkable insight, he discovered that the elliptic integral  $G(a, b)$  in (9) remains invariant if the parameters  $(a, b)$  are replaced with their arithmetic and geometric means; that is,

$$G(a, b) = G\left(\frac{a+b}{2}, \sqrt{ab}\right). \quad (12)$$

Iterating, passing to the limit, and using the invariance of the elliptic integral  $G$  yields

$$G(a, b) = \frac{\pi}{2 \operatorname{AGM}(a, b)}. \quad (13)$$

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<sup>1</sup>The authors learned of this quote from the preface of [5].

The convergence of the arithmetic-geometric mean iteration (11) is quadratic, meaning that  $|a_{n+1} - \text{AGM}(a, b)| \leq C|a_n - \text{AGM}(a, b)|^2$  for some  $C > 0$ . Thus (11) leads to a rapid evaluation of the elliptic integral  $G(a, b)$ . This iteration has been used for the numerical evaluation of elliptic integrals. See [7], [8], [9], [10], or [11] for details. The algorithm described here could also be used for the numerical evaluation of rational integrals.

It was a pleasant surprise when, in the process of analyzing definite integrals of rational functions, we discovered that

$$U_6 = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx$$

admits a similar invariant transformation. We call this a *rational Landen transformation*. In the case of  $U_6$  the dynamical system (11) is replaced with

$$\begin{aligned} a_{n+1} &= \frac{a_n b_n + 5a_n + 5b_n + 9}{(a_n + b_n + 2)^{4/3}}, \\ b_{n+1} &= \frac{a_n + b_n + 6}{(a_n + b_n + 2)^{2/3}}, \end{aligned} \tag{14}$$

with similar rules for  $c_n$ ,  $d_n$ , and  $e_n$ . The derivation of (14) appears in [2].

The sequence  $(a_n, b_n)$  converges to  $(3, 3)$  precisely for those initial data  $(a_0, b_0)$  for which the integral  $U_6$  is finite. Moreover, for the numerator parameters, we have  $(c_n, d_n, e_n) \rightarrow (1, 2, 1)L$ , for some real  $L$ . The convergence of this method is discussed in [2], [12], and [14]. The invariance of  $U_6$  yields the identity

$$U_6 = \frac{\pi}{2L}$$

exactly as in (13). Observe that (6) is also of this type: an integral given as the limit of an iterative process. Transformations similar to (14) have been produced in [3] for any even rational integrand.

Until now all rational Landen transformations were restricted to even rational functions. In this paper we present the simplest example of a technique that we expect will extend to the general case (see [17] for details).

The identities (6) and (13) yield *iterative methods* to evaluate the corresponding integrals. For example, the first four iterations of the evaluation of

$$I = \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 3x + 1}$$

using (3) are given in Table 1.

Table 1.

$n$	$a_n$	$b_n$	$c_n$
0	4	3	1
1	1.0731707317	0.6585365853	1.7317073171
2	1.3322738087	0.0186646386	1.31360991700
3	1.3228754233	$4.644065 \times 10^{-7}$	1.3228758877
4	1.3228756555	$7.154295 \times 10^{-21}$	1.3228756555

The example presented in the table exhibits cubic convergence, faster than the convergence of the AGM. The exact value of  $I$  is  $2\pi/\sqrt{7}$ , and (6) yields  $\lim_{n \rightarrow \infty} a_n = \sqrt{7}/2$ . The reader can check that the value  $a_4$  gives  $\sqrt{7}/2$  correct to ten digits of accuracy.

At the end of the amazing numerical calculation that led him to establish the invariance for the elliptic integral  $G(a, b)$ , Gauss commented in his diary that this *will surely open up a whole new field of analysis*. This statement is certainly true. The reader will find in [5] a detailed discussion of how the arithmetic-geometric mean plays a fundamental role in modern computations of the digits of  $\pi$ . This technique has also been used in [4] to create new and efficient methods to evaluate elementary functions.

Over the years many proofs of (12) have been discovered. A number of them can be found in [18]. The authors are particularly fond of the succinct proof by D. J. Newman [19]: use  $x = b \tan \theta$  and follow with  $x \mapsto x + \sqrt{x^2 + ab}$ . *Change of variables is an art*.

**3. THE QUADRATIC CASE.** The goal of this section is to present the algebraic techniques that produce the transformation (3). We scale the integrand by multiplying both the numerator and denominator by an *appropriate* polynomial. This scaling is one of the main ingredients in the formulation of the Landen transformations. The other one will be discussed in the next section.

We are motivated by the identities

$$U(\tan \theta) = -\frac{\sin(3\theta)}{\cos^3 \theta} \quad , \quad V(\tan \theta) = -\frac{\cos(3\theta)}{\cos^3 \theta} ,$$

where

$$U(x) = x^3 - 3x \quad , \quad V(x) = 3x^2 - 1 .$$

The task is to find coefficients  $z_0, z_1, z_2, z_3, z_4$  and  $e_0, e_1, e_2$  such that

$$(ax^2 + bx + c)(z_0x^4 + z_1x^3 + z_2x^2 + z_3x + z_4) \tag{15}$$

can be written as

$$e_0U^2(x) + e_1U(x)V(x) + e_2V^2(x) \tag{16}$$

with unknown coefficients  $z_i$  and  $e_i$  that are functions of the original parameters  $a$ ,  $b$ , and  $c$ . *There is so much freedom, it can't be hard.*

Matching (15) with (16) yields a system of seven equations for the eight unknowns. We use the first five to solve for the coefficients  $z_i$  in terms of  $a$ ,  $b$ ,  $c$ , and the  $e_i$ . To start, comparison of the constant term in (15) and (16) gives

$$z_4 = c^{-1}e_2. \quad (17)$$

Using this value, we find that the first-order coefficient  $z_3$  satisfies  $cz_3 - 3e_1 + bc^{-1}e_2 = 0$ , which yields

$$z_3 = c^{-2}(3ce_1 - be_2). \quad (18)$$

The next powers produce

$$z_2 = c^{-3}(9c^2e_0 - 3bce_1 + b^2e_2 - ace_2 - 6c^2e_2)$$

and

$$z_1 = c^{-4}(-9bc^2e_0 + 3b^2ce_1 - 3ac^2e_1 - 10c^3e_1 - b^3e_2 + 2abce_2 + 6bc^2e_2),$$

respectively. Finally,

$$z_0 = c^{-5}(9b^2c^2e_0 - 9ac^3e_0 - 6c^4e_0 - 3b^3ce_1 + 6abc^2e_1 + 10bc^3e_1 + b^4e_2 - 3ab^2ce_2 + a^2c^2e_2 - 6b^2c^2e_2 + 6ac^3e_2 + 9c^4e_2).$$

This leaves the two equations that arise from the two highest powers, which we use to find the parameters  $e_i$ . We solve the  $x^5$  equation for  $e_2$  in terms of the parameters  $a$ ,  $b$ ,  $c$ ,  $e_1$ , and  $e_0$ . Substituting this information into the equation for the leading term produces

$$b(b^2 - 3(a - c)^2)e_0 = a(3b^2 - (a + 3c)^2)e_1. \quad (19)$$

The system has one degree of freedom, which we exploit to ensure that the  $z_i$  and  $e_i$  are polynomials in the parameters  $a$ ,  $b$ , and  $c$ . We initially choose  $e_0 = a((a + 3c)^2 - 3b^2)$ , from which it follows that  $e_1 = -b(b^2 - 3(a - c)^2)$ . This in turn yields  $e_2 = -c(3b^2 - (3a + c)^2)$ .

The expressions for the coefficients  $z_i$  reduce to the following:

$$\begin{aligned} z_0 &= (a + 3c)^2 - 3b^2 \\ z_1 &= 8b(a - 3c) \\ z_2 &= -6a^2 + 10b^2 + 44ac - 6c^2 \\ z_3 &= 8b(c - 3a) \\ z_4 &= (3a + c)^2 - 3b^2 \end{aligned}$$

and, just to reiterate,

$$\begin{aligned} e_0 &= a((a + 3c)^2 - 3b^2) \\ e_1 &= b(3(a - c)^2 - b^2) \\ e_2 &= c((3a + c)^2 - 3b^2). \end{aligned}$$

In the latter formulas we already see a semblance of the iteration (3).

**4. ENTER TRIGONOMETRY.** In this section we complete the construction of the Landen transformation and establish the invariance of the integral (1) under it. We establish the vanishing of a special class of integrals that appear as intermediate steps in this construction.

We start with (1) and use the change of variables  $x = \tan \theta$  to produce

$$I = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta}. \quad (20)$$

The identities

$$\tan^3 \theta - 3 \tan \theta = -\frac{\sin(3\theta)}{\cos^3 \theta}, \quad 3 \tan^2 \theta - 1 = -\frac{\cos(3\theta)}{\cos^3 \theta}$$

that were the reason behind the choices for  $U$  and  $V$  are then used to obtain

$$I = \sum_{k=0}^4 z_{4-k} \int_{-\pi/2}^{\pi/2} \frac{\sin^k \theta \cos^{4-k} \theta d\theta}{e_0 \sin^2(3\theta) + e_1 \sin(3\theta) \cos(3\theta) + e_2 \cos^2(3\theta)} \quad (21)$$

from the integral (1) after it has been scaled according to the procedure described in section 3.

The elementary identities

$$\begin{aligned} \cos^4 \theta &= \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8} \\ \cos^3 \theta \sin \theta &= \frac{1}{8} \sin(4\theta) + \frac{1}{4} \sin(2\theta) \\ \cos^2 \theta \sin^2 \theta &= \frac{1}{8} - \frac{1}{8} \cos(4\theta) \\ \cos \theta \sin^3 \theta &= \frac{1}{4} \sin(2\theta) - \frac{1}{8} \sin(4\theta) \\ \sin^4 \theta &= \frac{1}{8} \cos(4\theta) - \frac{1}{2} \cos(2\theta) + \frac{3}{8} \end{aligned} \quad (22)$$

transform the expression for  $I$  to a linear combination of

$$S_k = \int_{-\pi/2}^{\pi/2} \frac{\sin(k\theta) d\theta}{e_0 \sin^2(3\theta) + e_1 \sin(3\theta) \cos(3\theta) + e_2 \cos^2(3\theta)} \quad (k = 2, 4)$$

and

$$C_k = \int_{-\pi/2}^{\pi/2} \frac{\cos(k\theta) d\theta}{e_0 \sin^2(3\theta) + e_1 \sin(3\theta) \cos(3\theta) + e_2 \cos^2(3\theta)} \quad (k = 0, 2, 4)$$

The magic of the Landen transformations comes from the vanishing of many of these integrals. This reduces (21) to an integral of the type (20) with new coefficients, resulting in the transformation rule (3). Indeed, for even  $k$  the integrals  $S_k$  and  $C_k$  vanish if  $k$  is not a multiple of 3. To verify this, replace  $\theta$  with

$u = \theta + \pi$  in the definition of  $S_k$ . Using  $\sin(k[u - \pi]) = (-1)^k \sin(ku) = \sin(ku)$  and  $\cos(k[u - \pi]) = (-1)^k \cos(ku) = \cos(ku)$ , we arrive at

$$S_k = \int_{\pi/2}^{3\pi/2} \frac{\sin(ku) du}{e_0 \sin^2(3u) + e_1 \sin(3u) \cos(3u) + e_2 \cos^2(3u)}.$$

Adding this to the original  $S_k$  and taking advantage of the periodicity of the integrand we get

$$S_k = \frac{1}{2} \int_0^{2\pi} \frac{\sin(ku) du}{e_0 \sin^2(3u) + e_1 \sin(3u) \cos(3u) + e_2 \cos^2(3u)}.$$

Now, we observe that both  $\sin(3u)$  and  $\cos(3u)$  are invariant under shifts by  $2\pi/3$  and  $4\pi/3$ , so

$$6S_k = \int_0^{2\pi} \frac{\sin(ku) + \sin(ku - 2\pi k/3) + \sin(ku - 4\pi k/3)}{e_0 \sin^2(3u) + e_1 \sin(3u) \cos(3u) + e_2 \cos^2(3u)} du.$$

The numerator in the integrand is the imaginary part of

$$e^{iku} + e^{i(ku - 2\pi k/3)} + e^{i(ku - 4\pi k/3)} = e^{iku} \left( 1 + e^{-2\pi ki/3} + e^{-4\pi ki/3} \right),$$

and the last sum is 3 or 0 depending on whether 3 divides  $k$  or not.

We conclude that the only terms that contribute to (21) are the constants in (22). Therefore

$$I = \frac{1}{16} \int_0^{2\pi} \frac{3z_4 + z_2 + 3z_0}{e_0 \sin^2(3u) + e_1 \sin(3u) \cos(3u) + e_2 \cos^2(3u)} du,$$

where we have again appealed to periodicity to extend the integral to  $[0, 2\pi]$ . The change of variables  $\theta = 3u$  leads to

$$I = \frac{1}{8} \int_{-\pi/2}^{\pi/2} \frac{3z_4 + z_2 + 3z_0}{e_0 \sin^2 \theta + e_1 \sin \theta \cos \theta + e_2 \cos^2 \theta} d\theta,$$

so we have returned to the original form (20) but with different coefficients. The result in (4) is obtained by using  $x = \tan \theta$  and the following identities:

$$\begin{aligned} \frac{8e_0}{3z_4 + z_2 + 3z_0} &= a \left( \frac{(3a + c)^2 - 3b^2}{(3a + c)(a + 3c) - b^2} \right) \\ \frac{8e_1}{3z_4 + z_2 + 3z_0} &= b \left( \frac{3(a - c)^2 - b^2}{(3a + c)(a + 3c) - b^2} \right) \\ \frac{8e_2}{3z_4 + z_2 + 3z_0} &= c \left( \frac{(a + 3c)^2 - 3b^2}{(3a + c)(a + 3c) - b^2} \right). \end{aligned} \quad (23)$$

**5. THE ANALYSIS OF CONVERGENCE.** In the last two sections we have shown the invariance of (1) under the Landen transformation (3). We now



conclude by establishing the convergence of its iterates as in (5). In particular, we show that the error

$$e_n := (a_n - \frac{1}{2}\sqrt{4ac - b^2}, b_n, c_n - \frac{1}{2}\sqrt{4ac - b^2})$$

satisfies  $e_n \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover, we demonstrate cubic convergence:

$$\|e_{n+1}\| \leq C\|e_n\|^3 \quad (24)$$

for some positive constant  $C$ .

The analysis of convergence is simpler in the variables  $x = a + c$ ,  $y = b$ , and  $z = a - c$ . The dynamical system (3) translates to

$$\begin{aligned} x_{n+1} &= x_n \left[ \frac{4x_n^2 - 3z_n^2 - 3y_n^2}{4x_n^2 - y_n^2 - z_n^2} \right], \\ z_{n+1} &= z_n \left[ \frac{z_n^2 - 3y_n^2}{4x_n^2 - y_n^2 - z_n^2} \right], \\ y_{n+1} &= y_n \left[ \frac{3z_n^2 - y_n^2}{4x_n^2 - y_n^2 - z_n^2} \right], \end{aligned} \quad (25)$$

with initial conditions  $x_0 = x$ ,  $y_0 = y$ , and  $z_0 = z$ .

We now prove that

$$\lim_{n \rightarrow \infty} x_n = \sqrt{x^2 - y^2 - z^2}, \quad \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$$

or, equivalently, that

$$\lim_{n \rightarrow \infty} \left( x_n - \sqrt{x^2 - y^2 - z^2} \right)^2 + y_n^2 + z_n^2 = 0. \quad (26)$$

This is equivalent to (5), so it will finish the proof of convergence.

To complete the change of variables we use the invariance of the discriminant (7) to obtain

$$x_n^2 - y_n^2 - z_n^2 = x^2 - y^2 - z^2 = 4ac - b^2,$$

and we write  $w = \sqrt{4ac - b^2}$ . The first equation of iteration (25) becomes

$$x_{n+1} = x_n \left[ \frac{x_n^2 + 3w^2}{3x_n^2 + w^2} \right], \quad (27)$$

with initial conditions  $x_0 = a + c (> 0)$ . The required limit in (26) is now

$$\lim_{n \rightarrow \infty} x_n(x_n - w) = 0, \quad (28)$$

with  $y_n$  and  $z_n$  absorbed into the constant  $w$ . *The number of variables has been reduced from three to one.*

Using one last change of variables,  $q_n = -ix_n/w$ , we reduce (27) to

$$q_{n+1} = \frac{q_n^3 - 3q_n}{3q_n^2 - 1} = \frac{U(q_n)}{V(q_n)}. \quad (29)$$

What we need to prove in order to establish (28) is that  $q_n \rightarrow -i$ . (The polynomials  $U$  and  $V$  introduced in section 3 have miraculously reappeared!) The trigonometric identity

$$\frac{U(\cot \theta)}{V(\cot \theta)} = \cot(3\theta),$$

coupled with a representation of the initial condition as

$$q_0 = \cot(it) \quad (30)$$

for some  $t$  ( $0 < t < \infty$ ), shows that (29) simplifies to

$$q_1 = \frac{U(\cot it)}{V(\cot it)} = \cot(3it)$$

and, in general,

$$q_n = \cot(3^n it) = -i \frac{e^{2t3^n} + 1}{e^{2t3^n} - 1}.$$

We conclude that  $q_n \rightarrow -i$ , whence  $x_n \rightarrow w$ , as desired.

To verify (30), we write  $q_0 = -id$  with  $d = (a + c)/\sqrt{4ac - b^2}$ . Now recall that  $4ac - b^2 > 0$ . An elementary argument shows that  $d \geq 1$ , so we can take

$$t = \coth^{-1}(d) = \frac{1}{2} \ln \frac{d+1}{d-1}.$$

The fact that the convergence is cubic follows directly from

$$|q_n + i| = \frac{2}{e^{2t3^n} - 1},$$

which decreases to 0 like  $e^{-2t3^n}$ . This implies (24) and completes the proof of convergence.

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**DANTE MANNA** received his Bachelor of Arts degree in mathematics from Wesleyan University in Middletown, Connecticut, in May 2001. He was also a Spring 2000 graduate of the Associated Colleges in China intensive Mandarin program, at Capital University in Beijing, China. He has been enrolled in the Ph.D. program at Tulane University since August 2001 and will graduate in May 2006 as a student of Victor H. Moll.

**VICTOR H. MOLL** is a professor of mathematics at Tulane University. He studied under Henry McKean at the Courant Institute of New York University. After graduation in 1984, he spent two years at Temple University in Philadelphia. There he failed to pay attention when Donald J. Newman was telling him about some transformations on elliptic integrals. His research interests are in the mathematics behind the evaluation of definite integrals.

*Department of Mathematics, Tulane University, New Orleans, LA 70118*

*dmanna@math.tulane.edu and vhm@math.tulane.edu*