Arithmetic properties of plane partitions

To Doron: a wonderful Mensch

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Abstract

The 2-adic valuations of sequences counting the number of alternating sign matrices of size n and the number of totally symmetric plane partitions are shown to be related in a simple manner.

Keywords: valuations, alternating sign matrices, totally symmetric plane partitions.

1 Introduction

A *plane partition* (PP) is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that π has finite support and is weakly decreasing in rows and columns. These partitions are often represented by solid Young diagrams in 3-dimensions. MacMahon found a complicated formula for the enumeration of all PPs inside an n-cube. This was later simplified to

$$\texttt{PP}_n = \prod_{i,j,k=1}^{n} \frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

A plane partition is called *symmetric* (SPP) if $\pi_{ij} = \pi_{ji}$ for all indices i, j. The number of such partitions whose solid Young diagrams fit inside an n-cube is given by

$$\texttt{SPP}_n = \prod_{j=1}^{n} \prod_{i=1}^{n} \frac{i+j+n-1}{i+j+i-2} = \prod_{j=1}^{n} \prod_{i=j}^{n} \frac{i+j+n-1}{i+j-1}. \tag{2}$$

Another interesting subclass of partitions is that of *totally symmetric plane partitions* (TSPP). These are symmetric partitions π such that every row of π is self-conjugate as an ordinary plane partition (or the Young diagrams are invariant under any permutation of the axes). J. Stembridge [3] showed that the number of TSPP in an n-cube is given by

$$\mathtt{TSPP}_n = \prod_{1\le i\le j\le k\le n} \frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{n}\prod_{i=j}^{n} \frac{i+j+n-1}{i+j+i-2} = \prod_{1\le i\le j\le n} \frac{i+j+n-1}{i+j+j-2}. \tag{3}$$

For the solid Young diagram of a plane partition π that fits inside a box of a given size, one can take the collection of cubes that are in the box but *do not* belong to the solid Young diagram. These determine another plane partition called the *complement* of π. If the complement of π is the same as the original partition, π is called *self-complementary*. Such partitions only fit in an even-dimensional box. The number of plane partitions inside a $2n \times 2n \times 2n$ box that are both totally symmetric and self-complementary (TSSCPP) is given by

$$\mathtt{TSSCPP}_{2n} = \prod_{1\le i\le j\le n} \frac{i+j+n-1}{i+j+i-1}. \tag{4}$$

The proof required the efforts of three combinatorialists: W. F. Doran, J. Stembridge and G. Andrews.

An *alternating sign matrix* (ASM) is an array of 0, 1 and -1 such that the entries of each row and column add up to 1 and the non-zero entries of a given row/column alternate. After a fascinating sequence of events, D. Zeilberger [5] completely proved the conjecture that the number of ASM of size n equals $\mathtt{TSSCPP}_{2n}$. Bressoud's book [1] contains an entertaining story of these counting functions.

Note. For simplicity, we write $A_n = \mathtt{TSSCPP}_{2n}$, $B_n = \mathtt{TSPP}_n$ and $T_n = \mathtt{PP}_n$.

A simple calculation shows that A_n and B_n do not divide each other as integers. The first few values of the quotient A_n/B_n are given by

$$\frac{1}{2}, \frac{2}{5}, \frac{7}{16}, \frac{7}{11}, \frac{39}{32}, \frac{52}{17}, \frac{3211}{320}, \frac{988}{23}, \frac{30685}{128}, \frac{50540}{29}. \tag{5}$$

The quotient A_n/B_n presents a large amount of cancellation. For instance, the integers A_{40}, B_{40} have 182 and 100 digits and the reduced form of A_n/B_n has denominator 17. Motivated by this cancellation, during a conference in the summer of 2010 at Nankai University, where Manuel Kauers explained the remarkable result [2], one of the authors computed a list of the values when B_n is odd. This question had also been the key to the main ideas behind the arithmetic properties of A_n, as described in [4]. Figure 1 depicts the 2-adic valuation of A_n.

The computation showed that the indices where B_{2n} is odd is related to the *Jacobsthal numbers* that are defined by the recurrence $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 1$ and $J_1 = 1$. These are precisely the indices where A_n is odd. This observation lead to the first result in this paper.

Note. For $n \in \mathbb{N}$, denote by $\nu_2(n)$ the 2-adic valuation of n, defined as the highest power of 2 that divides n. Let $s_2(n)$ equal to the sum of the binary digits of n.

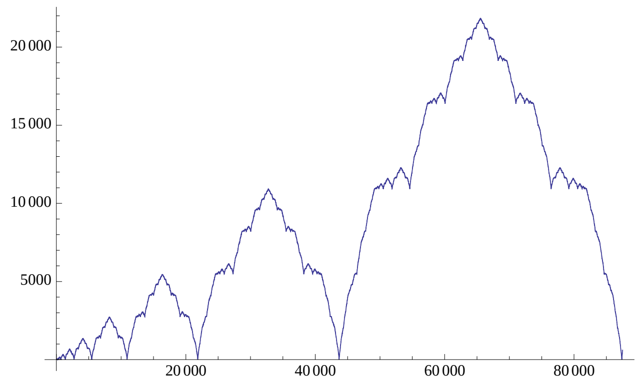

Figure 1: The 2-adic valuation of A_n

Theorem 1.1 *For $n \in \mathbb{N}$. Then,*

$$\begin{aligned} \nu_2(B_{2n}) &= \nu_2(A_n) \\ \nu_2(B_{2n-1}) &= \nu_2(A_n) + 2n - 1. \end{aligned}$$

Proof. To compare A_n with B_{2n}, compute the ratios

$$\begin{aligned} \frac{A_{n+1}}{A_n} &= \prod_{j=1}^{n+1}\prod_{i=1}^{j}\frac{i+j+n}{i+j+i-1}\prod_{j=1}^{n}\prod_{i=1}^{j}\frac{i+j+i-1}{i+j+n-1} \\ &= \frac{3n+2}{n+1}\prod_{i=1}^{n}(i+2n+1)(i+2n)\prod_{j=1}^{n-1}\frac{1}{2j+n+1}\prod_{i=1}^{n+1}\frac{1}{i+i+n} \\ &= \prod_{i=1}^{n}\frac{(i+2n+1)(i+2n)}{(2i+n-1)(2i+n)} \end{aligned}$$

and

$$\begin{aligned} \frac{B_{2n+2}}{B_{2n}} &= \prod_{k=1}^{2n+2}\prod_{j=1}^{k}\prod_{i=1}^{j}\frac{i+j+k-1}{i+j+k-2}\prod_{k=1}^{2n}\prod_{j=1}^{k}\prod_{i=1}^{j}\frac{i+j+k-2}{i+j+k-1} \\ &= \prod_{j=1}^{2n+1}\prod_{i=1}^{j}\frac{i+j+2n}{i+j+2n-1}\prod_{j=1}^{2n+2}\prod_{i=1}^{j}\frac{i+j+2n+1}{i+j+2n} \\ &= \frac{(6n+1)(6n+3)(6n+5)}{(2n+1)(2n+2)(2n+3)}\prod_{i=1}^{2n-1}\frac{(i+4n+1)(i+4n+3)}{(2i+2n+2)(2i+2n+3)} \\ &= \frac{(6n+5)}{(2n+1)}\prod_{i=1}^{2n}\frac{(i+4n+1)(i+4n+3)}{(2i+2n)(2i+2n+1)} \\ &= \frac{(6n+5)}{(2n+1)}\frac{\prod_{i=1}^{n}(2i+4n+1)(2i+4n+3)}{\prod_{i=1}^{2n}(2i+2n+1)}\frac{\prod_{i=1}^{n}(2i+4n)(2i+4n+2)}{\prod_{i=1}^{2n}(2i+2n)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(6n+5)}{(2n+1)} \prod_{i=1}^{n} \frac{(2i+4n+1)(2i+4n+3)}{(4i+2n+1)(4i+2n-1)} \prod_{i=1}^{n} \frac{(2i+4n)(2i+4n+2)}{(4i+2n)(4i+2n-2)} \\
&= \frac{(6n+5)}{(2n+1)} \prod_{i=1}^{n} \frac{(2i+4n+1)(2i+4n+3)}{(4i+2n+1)(4i+2n-1)} \prod_{i=1}^{n} \frac{(i+2n)(i+2n+1)}{(2i+n)(2i+n-1)} \\
&= \frac{(6n+5)}{(2n+1)} \prod_{i=1}^{n} \frac{(2i+4n+1)(2i+4n+3)}{(4i+2n+1)(4i+2n-1)} \times \frac{A_{n+1}}{A_n}.
\end{aligned}
$$

Since $\nu_2(B_2) = \nu_2(A_1) = 0$ and $\nu_2(B_{2n+2}) - \nu_2(B_{2n}) = \nu_2(A_{n+1}) - \nu_2(A_n)$, the first assertion follows. Similarly,

$$
\begin{aligned}
\frac{B_{2n+1}}{B_{2n}} &= \prod_{i=1}^{n+1} \frac{(2i+4n+1)(2i+2n)}{(4i+2n-1)(4i+2n-3)} \times \frac{A_{n+1}}{A_n} \qquad (6) \\
&= 2^{n+1} \frac{(2n+1)!}{n!} \prod_{i=1}^{n+1} \frac{(2i+4n+1)}{(4i+2n-1)(4i+2n-3)} \times \frac{A_{n+1}}{A_n}.
\end{aligned}
$$

Hence

$$
\nu_2(B_{2n+1}) - \nu_2(B_{2n}) = n+1+2n+1-s_2(2n+1)-n+s_2(n)+\nu_2(A_{n+1})-\nu_2(A_n), \quad (7)
$$

where Legendre's formula $\nu_2(m!) = m - s_2(m)$ is applied. The rest follows from $s_2(2n+1) = s_2(n)+1$ and the first part of the proof.

2 A product identity

In this section we consider the function $\mathtt{SPP}_n$ counting the number of symmetric plane partitions of size n. Recall

$$
\mathtt{SPP}_n = \prod_{j=1}^{n} \prod_{i=1}^{n} \frac{i+j+n-1}{i+j+i-2}. \qquad (8)
$$

The next result appears to be new and is similar to

$$
\texttt{cylindrically symmetric} = (\texttt{totally symmetric})^2.
$$

Theorem 2.1 *The identity $\mathtt{SPP}_n = \mathtt{TSSCPP}_{2n} \times \mathtt{TSPP}_n$ holds.*

Proof: After some regrouping and re-indexing,

$$
\begin{aligned}
\mathtt{TSSCPP}_{2n} &= \prod_{j=1}^{n} \prod_{i=1}^{j} \frac{i+j+n-1}{i+j+i-1} \\
&= \prod_{j=1}^{n} \prod_{i=1}^{j} (i+j+n-1) \prod_{j=2}^{n} \prod_{i=1}^{j-1} (i+j+i-2)^{-1} \prod_{i=1}^{n} (2i+n-1)^{-1},
\end{aligned}
$$

and

$$\begin{aligned}
\mathtt{TSPP}_n &= \prod_{j=1}^{n}\prod_{i=j}^{n}\frac{i+j+n-1}{i+j+i-2}\\
&= \prod_{j=1}^{n}\prod_{i=j+1}^{n}(i+j+n-1)\prod_{j=2}^{n}\prod_{i=j}^{n}(i+j+i-2)^{-1}\prod_{j=1}^{n}(2j+n-1)\prod_{i=1}^{n}(2i-1)^{-1}.
\end{aligned}$$

Combining the two it follows that

$$\begin{aligned}
\mathtt{TSSCPP}_{2n}\times\mathtt{TSPP}_n &= \prod_{j=1}^{n}\prod_{i=1}^{n}(i+j+n-1)\prod_{j=2}^{n}\prod_{i=1}^{n}(i+j+i-2)^{-1}\prod_{i=1}^{n}(2i-1)^{-1}\\
&= \prod_{j=1}^{n}\prod_{i=1}^{n}(i+j+n-1)\prod_{j=1}^{n}\prod_{i=1}^{n}(i+j+i-2)^{-1}\\
&= \prod_{j=1}^{n}\prod_{i=1}^{n}\frac{i+j+n-1}{i+j+i-2}\\
&= \mathtt{SPP}_n.
\end{aligned}$$

The next statement follows from Theorem 1.1 and Theorem 2.1.

Corollary 2.2 *For $n\in\mathbb{N}$,*

$$\nu_2(SPP_{2n}) = \nu_2(A_{2n}) + \nu_2(A_n) \tag{9}$$

and

$$\nu_2(SPP_{2n-1}) = \nu_2(A_{2n-1}) + \nu_2(A_n) + 2n-1. \tag{10}$$

3 Some conjectures

This last section contains some conjectures. The first one deals with the 2-adic valuation of the sequences B_n and T_n.

Conjecture 3.1 *For $n\in\mathbb{N}$, the inequalities*

$$\nu_2(T_{2n}) > \nu_2(B_{2n}) \text{and} \nu_2(\mathrm{T}_{2n+1}) < \nu_2(\mathrm{B}_{2n+1}) \tag{11}$$

hold.

The second conjecture is related to sequences formed by succesive ratios.
Given a sequence of positive numbers $\{a_n\}$ consider the succesive ratios defined by

$$a_{n+1}^{\{0\}} := a_{n+1} \text{and} \mathrm{a}_{n+1}^{\{k\}} := \mathrm{a}_{n+1}^{\{k-1\}}/\mathrm{a}_{n}^{\{k-1\}}. \tag{12}$$

For instance,

$$a_{n+1}^{\{1\}} = \frac{a_{n+1}}{a_n} \text{and} \mathrm{a}_{\mathrm{n+1}}^{\{2\}} = \frac{\mathrm{a}_{\mathrm{n+1}}\mathrm{a}_{\mathrm{n-1}}}{\mathrm{a}_{\mathrm{n}}^2}. \tag{13}$$

In particular a_n is nonincreasing if $a_{n+1}^{\{1\}} \leq 1$ and *logconcave* if $a_{n+1}^{\{2\}} \leq 1$ and *logconvex* if $a_{n+1}^{\{2\}} \geq 1$.

Conjecture 3.2 *Let A_n be the ASM sequence. For $0 \leq k \leq 3$ the iterated sequence $A_{n+1}^{\{k\}}$ is logconvex. For $k \geq 4$, the sequence $A_{n+1}^{\{k\}}$ is logconvex when k is odd and logconcave when k is even.*

Problem 3.3 *Find a combinatorial proof of Theorem 2.1.*

Note. The calculations were performed after the talk. There were no violations to the ***Zeilberger rules of order***.

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