Bernoulli on Arc Length

VICTOR MOLL
Tulane University
New Orleans, Louisiana 70148
vhm@math.tulane.edu

JUDITH NOWALSKY
University of New Orleans
New Orleans, Louisiana 70148
jnowalsk@math.uno.edu

GINED ROA
LEONARDO SOLANILLA
Universidad del Tolima
Ibagué, Colombia
gined@latinmail.com
solanila@bunde.tolinet.com.co

The academic life of the Bernoulli family was always surrounded by controversy. The disputes between Johann (John) and his older brother and former teacher Jacob and with his son Daniel are famous and well documented. An interesting discussion of this remarkable family is found in Section 12.6 of [**3**]. After the death of L'Hôpital, John claimed the authorship of his classical analysis book. In the controversy between Leibniz and Newton about the creation of calculus, he stood on Leibniz' side. His controversial positions were not restricted to mathematics: he was even accused of denying the possibility of the resurrection of Christ.

In the course of our study of the history of elliptic integrals, we found a paper by Johann Bernoulli [**1**] which, in our opinion, both illuminates the calculation of arc lengths of smooth curves, a topic covered in most undergraduate calculus programs around the world, and provides an additional tool for producing new and interesting examples of *rectifiable curves*. According to Bernoulli, these are curves whose arc length can be expressed as elementary functions of their end points. The paper contains a main theorem that is perfectly valid even today, and admits a nice interpretation in terms of the notion of radius of curvature. Furthermore, we discovered in it a colorful antecedent of Landen integral transformations [**2**].

Let $y = y(x)$ be a differentiable function defined on $[a, b]$. Then its *arc length* is defined by

$$g(x) = \int_a^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi.$$

In general, this integral is not trivial. The examples and exercises provided in most textbooks look *unnatural*: for instance, the first example given in Thomas [**4**], page 395, deals with the arc length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

for $0 \leq x \leq 1$. This is an easy example in the sense that it is computable:

$$g(1) = \int_0^1 \sqrt{1 + 8\xi}\, d\xi = \frac{13}{6}.$$

The reader can easily verify that the integral corresponding to the length of a circle can be evaluated. However, the calculation of the arc length of an ellipse leads to the integral

$$L(x) = a\int_0^x \sqrt{\frac{1-e^2\xi^2}{1-\xi^2}}\, d\xi,$$

where a is the semimajor axis of the ellipse, and e its eccentricity. This last integral is one of the fundamental *elliptic integrals* and is not an elementary function. It was the starting point of our research on Bernoulli's work.

Bernoulli's *universal theorem* The main goal of this section is to present Bernoulli's result on how to produce rectifiable curves, which in this sense might also be called *rectifiable by straight lines*; their arc length can be expressed as elementary functions of their end points.

THEOREM 1. *Let $y = y(x)$ be a twice differentiable function satisfying*

$$\left(\frac{dy}{dx}\right)^2 + 3x\frac{dy}{dx}\frac{d^2y}{dx^2} \geq 0 \quad (\leq 0)$$

in its interval of definition $[a, b]$. Define a new curve with coordinates

$$X = x\left(\frac{dy}{dx}\right)^3, \quad Y = \frac{3x}{2}\left(\frac{dy}{dx}\right)^2 - \frac{1}{2}\int_a^x \left(\frac{dy}{d\xi}\right)^2 d\xi.$$

Now let $g(x)$ and $G(x)$ be the arc lengths of y and the parametric curve $(X(x), Y(x))$ starting at $x = a$. Then

$$g(x) + (-)\, G(x) = \xi\left(\frac{dg}{d\xi}\right)^3\Bigg]_{\xi=a}^{\xi=x}$$

for all $x \in [a, b]$.

Proof. First observe that

$$\left(\frac{dg}{dx}\right)^3 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$$

Following Bernoulli's recommendation, we compute

$$\frac{d}{dx}x\left(\frac{dg}{dx}\right)^3 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} + 3x\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}\frac{dy}{dx}\frac{d^2y}{dx^2}.$$

On the other hand, careful differentiation shows that

$$\begin{aligned} G(x) &= \int_a^x \sqrt{\left(\frac{dX}{d\xi}\right)^2 + \left(\frac{dY}{d\xi}\right)^2}\, d\xi \\ &= \int_a^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2}\left|\left(\frac{dy}{d\xi}\right)^2 + 3x\frac{dy}{d\xi}\frac{d^2y}{d\xi^2}\right| d\xi. \end{aligned}$$

To conclude the proof, note that the integrand of $g(x) + (-)\, G(x)$ is $\frac{d}{dx}x(\frac{dg}{dx})^3$, so the result follows from the Fundamental Theorem of Calculus. ∎

For example, the function $y = \ln x$ yields

$$X = \frac{1}{x^2}, \quad Y = \frac{2}{x^2} - \frac{1}{2} = 2\sqrt{X} - \frac{1}{2}.$$

After struggling to get the correct constants in some assertions in Bernoulli's article, we discovered a nice interpretation of Theorem 1. This formulation eluded Bernoulli as he did not relate the result to the curvature of the graph $y = y(x)$. Recall that the radius of curvature of the graph $y = y(x)$ at a point x is

$$R(x) = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \times \left(\frac{d^2y}{dx^2}\right)^{-1}.$$

This is the radius of a circle whose curvature matches that of the curve at the given point. Let us restate the previous theorem in terms of curvature.

THEOREM 2. *Under the assumptions of Theorem 1,*

$$g(x) \pm G(x) = \xi \frac{d^2y}{d\xi^2} R(\xi)\Big]_{\xi=a}^{\xi=x}.$$

Geometrically, denote respectively by $C(a), C(x)$ the centers of the osculating circles at the points $A(a) = (a, y(a)), A(x) = (x, y(x))$ on the curve. Also, let $\alpha(a) = \angle BAC(a), \alpha(x) = \angle BAC(x)$ be the corresponding angles between the radii of curvature $R(a) = CA(a), R(x) = CA(x)$ at these points and the hypotenuses of some right triangles $ABC(a), ABC(x)$ as sketched as in the figure below. The positions of points $B(a)$ and $B(x)$ along the rays shown are determined by the angles $\alpha(a)$ and $\alpha(x)$, respectively. Then

$$CB(a) = R(a)\tan\alpha(a), \quad CB(x) = R(x)\tan\alpha(x).$$

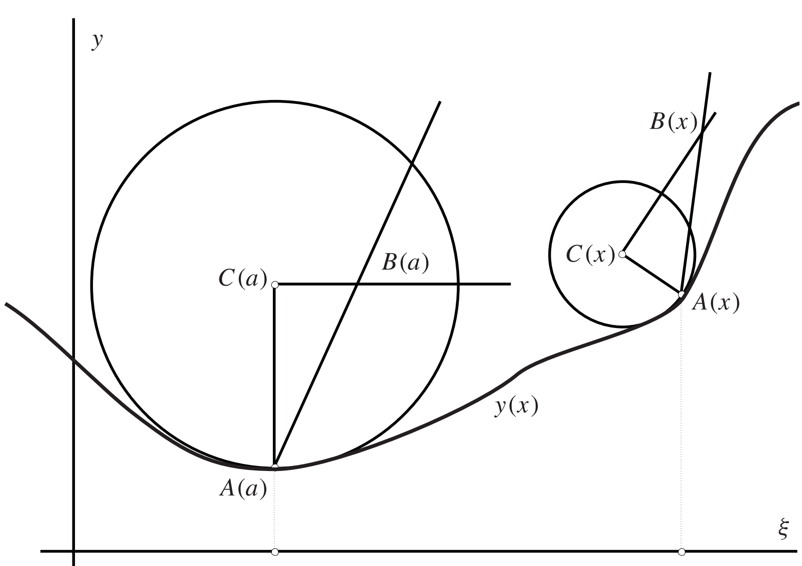

Figure 1 Geometric interpretation

In order to get the rectification, set

$$\tan\alpha(a) = a\frac{d^2y}{d\xi^2}\bigg]_{\xi=a}, \qquad \tan\alpha(x) = x\frac{d^2y}{d\xi^2}\bigg]_{\xi=x}.$$

In this way,

$$g(x) \pm G(x) = CB(x) - CB(a)$$

gives a new meaning to Theorem 2: the sum (difference) of the arc length integrals equals the difference of two straight segments. Bernoulli was proud to declare that this sum (difference) could be measured on a *straight line*.

Parabolas In Bernoulli's language, a *parabola* is a curve defined by the function $y = x^q$, for q a rational number. In this section we discuss parabolas that are rectifiable by the above method. Remember that a curve $y = y(x)$ is rectifiable if its arc length integral admits an antiderivative in terms of elementary functions. Bernoulli was interested in the question of rectifiable parabolas and was aware of the following result.

THEOREM 3. *Let n be a nonzero integer. Then the parabola $y = x^{\frac{2n+1}{2n}}$ is rectifiable on* [0, 1].

Proof. The arc length is

$$g(x) = \int_0^x \sqrt{1 + \left(\frac{2n+1}{2n}\right)^2 \xi^{1/n}}\, d\xi,$$

and the substitution $u(\xi) = 1 + (\frac{2n+1}{2n})^2\xi^{1/n}$ yields

$$g(x) = n\left(\frac{2n}{2n+1}\right)^{2n} \int_1^{u(x)} \sqrt{u}(u-1)^{n-1}\, du,$$

which can be evaluated by expanding $(u-1)^{n-1}$ using the binomial theorem. ■

The reader may recognize that this result is the source of most arc length exercises in textbooks. Our first example corresponds to $n = 1$. Moreover, the presence of the factor $4\sqrt{2}/3$ is not essential to the solution of the problem: it is window dressing.

We can now use Theorem 1 to assert that every parabola can be rectified by adding the arc length of another (conveniently chosen) parabola.

THEOREM 4. *Any parabola $y = x^q$, $q \neq 2/3$, can be rectified by adding (subtracting) to its arc length the arc length of the auxiliary parabola*

$$Y = \frac{3q-2}{2q-1} q^{\frac{1}{2-3q}} X^{\frac{2q-1}{3q-2}},$$

where $X = q^3x^{3q-2}$. In particular, the usual quadratic (Archimedean) parabola $y = x^2$ is rectified by adding the arc length of the biquadratic-cubic parabola $Y = \frac{4}{3}2^{-1/4}X^{3/4}$.

Proof. Note that

$$\left(\frac{dy}{dx}\right)^2 + 3x\frac{dy}{dx}\frac{d^2y}{dx^2} = (3q-2)q^2x^{2(q-2)}.$$

The rest of the proof is a straightforward calculation. ■

Integral transformations Many interesting questions can be formulated at this point. For instance: *Under what circumstances does the degree of the auxiliary parabola equal the degree of the original parabola?* The answer is clearly given by the fixed points of the rational transformation $b(q) = (2q-1)/(3q-2),\ q \neq 2/3$, namely, $q = 1, 1/3$. Since the first value gives a trivial answer, Bernoulli considered only the second value, which corresponds to the *primary cubic* parabola $y = x^{1/3}$. This case is important because it yields $Y = X^{1/3}$, and consequently

$$g(x) - G(x) = \int_0^x \sqrt{1 + \frac{1}{9\xi^{4/3}}}\, d\xi - \int_0^{\frac{1}{27x}} \sqrt{1 + \frac{1}{9X^{4/3}}}\, dX$$

$$= \int_{\frac{1}{27x}}^x \sqrt{1 + \frac{1}{9\xi^{4/3}}}\, d\xi = x\left(1 + \frac{1}{9x^{4/3}}\right)^{3/2},$$

an actual arc length integral formula!

It is interesting to study the sequence defined recursively by $p_0 = q,\ p_n = bp_{n-1}$, $n = 1, 2, \ldots$, for a given starting value q. For example, if $q = 2$, then $p_1 = 3/4$ and $p_2 = 2$ (again). This implies that if the original parabola is the usual $y = x^2$, for which $X = 8x^7$ and $Y = \frac{4}{3}2^{-1/4}X^{3/4}$, then

$$\int_0^x \sqrt{1 + 4\xi^2}\, d\xi + \int_0^{8x^7} \sqrt{1 + \frac{1}{\sqrt{2X}}}\, dX = x(1 + 4x^2)^{3/2}.$$

But applying the transformation from Theorem 1 once more to the auxiliary parabola, we obtain $\mathcal{X} = 2^{-3/4}X^{1/4} = x^{7/4}$ and $\mathcal{Y} = 2^{-3/2}\sqrt{X} = \mathcal{X}^2$. Thus

$$\int_0^{8x^7} \sqrt{1 + \frac{1}{\sqrt{2X}}}\, dX + \int_0^{x^{7/4}} \sqrt{1 + 4\mathcal{X}^2}\, d\mathcal{X} = 8x^7\left(1 + \frac{1}{4x^{7/2}}\right)^{3/2},$$

from which

$$\int_x^{x^{7/4}} \sqrt{1 + 4\xi^2}\, d\xi = 8x^7\left(1 + \frac{1}{4x^{7/2}}\right)^{3/2} - x(1 + 4x^2)^{3/2}.$$

Finally, we may ask these questions: *How many different values can a sequence p_n take and still lead to an arc length integral formula? Is there any relation between the convergence of this type of sequence and new arc length integral formulas?*

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