THE EVALUATION OF A QUARTIC INTEGRAL VIA SCHWINGER, SCHUR AND BESSEL

TEWODROS AMDEBERHAN, VICTOR H. MOLL, AND CHRISTOPHE VIGNAT

Abstract. We provide additional methods for the evaluation of the integral

$$N_{0,4}(a;m) \quad := \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

where $m \in \mathbb{N}$ and $a \in (-1,\infty)$ in the form

$$N_{0,4}(a;m) \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a)$$

where $P_m(a)$ is a polynomial in a. The first one is based on a method of Schwinger to evaluate integrals appearing in Feynman diagrams, the second one is a byproduct of an expression for a rational integral in terms of Schur functions. Finally, the third proof, is obtained from an integral representation involving modified Bessel functions.

1. Introduction

The definite integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1.1}$$

with $m \in \mathbb{N}$ and $a > -1$ has the value

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a) \tag{1.2}$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k. \tag{1.3}$$

The reader will find in [3] a survey of the many proofs of (1.2) that have appeared in the literature. After this survey was written, other proofs have appeared; see [5, 6, 8, 10]. The goal of this note is to present some additional proofs: the first one is based in the *Schwinger parametrization*, a second one using *Schur functions* and the last one involves a representation involving integrals of *Bessel functions*.

Date: July 27, 2010.

2000 *Mathematics Subject Classification.* Primary 33C05, 33C10, 33F10.

Key words and phrases. integrals, hypergeomteric functions, Bessel functions, Schur functions, WZ method.

2. Schwinger parametrization

The method of Schwinger parameters, described in classical quantum theory books such as [15], is utilized here to give a proof of (1.2). A preliminary discussion is presented first.

Starting from

$$\frac{1}{A_1A_2} = \int_0^1 \frac{dx}{\left(xA_1+(1-x)A_2\right)^2} = \int_0^1\int_0^1 \frac{\delta\left(1-x_1-x_2\right)}{\left(x_1A_1+x_2A_2\right)^2}dx_1dx_2 \tag{2.1}$$

where δ is the Dirac measure, it can be deduced by induction that

$$\frac{1}{\prod_{i=1}^n A_i} = (n-1)!\int_{[0,1]^n} \frac{\delta\left(1-\sum_{i=1}^n x_i\right)}{\left(\sum_{i=1}^n x_iA_i\right)^n}\prod_{i=1}^n dx_i. \tag{2.2}$$

The induction step uses the n-times differentiated version of (2.1), namely

$$\frac{1}{A_1A_2^{\nu_2}} = \int_0^1\int_0^1 \frac{\delta\left(1-x_1-x_2\right)\nu_2 x_2^{\nu_2-1}}{\left(x_1A_1+x_2A_2\right)^{\nu_2+1}}dx_1dx_2. \tag{2.3}$$

Repeated differentiations of (2.2) yields the more general result

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\nu\right)}{\prod_{i=1}^n\Gamma\left(\nu_i\right)}\int_{[0,1]^n} \frac{\delta\left(1-\sum_{i=1}^n x_i\right)\prod_{i=1}^n x_i^{\nu_i-1}\,dx_i}{\left(\sum_{i=1}^n x_iA_i\right)^{\nu}} \tag{2.4}$$

with $\nu=\nu_1+\cdots+\nu_n$ and $A_i\in\mathbb{C}$.

An alternative proof has been provided by Shapiro in his lecture notes [17]. This direct proof of (2.2) starts with

$$\frac{1}{A} = \int_0^\infty e^{-Ax}dx \tag{2.5}$$

for $\operatorname{Re} A>0$. It follows that

$$\prod_{i=1}^n \frac{1}{A_i} = \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n A_ix_i}\prod_{i=1}^n dx_i \tag{2.6}$$

Now let $x=x_1+\cdots x_n$ and $y_i=\frac{x_i}{x}$ to obtain

$$\prod_{i=1}^n dx_i = x^{n-1}dx\prod_{i=1}^n \delta\left(1-\sum_{i=1}^n y_i\right)\, dy_i \tag{2.7}$$

so that

$$\prod_{i=1}^n \frac{1}{A_i} = \int_{\mathbb{R}_+^n}\prod_{i=1}^n \delta\left(1-\sum_{i=1}^n y_i\right)x^{n-1}dx\, e^{-x\sum_{i=1}^n A_iy_i}\,dy_i. \tag{2.8}$$

After a change of variable, the latter integral is evaluated as

$$\int_0^\infty x^{n-1}e^{-x\sum_{i=1}^n A_iy_i}dx = \Gamma(n)\left(\sum_{i=1}^n A_iy_i\right)^{-n} \tag{2.9}$$

resulting in

$$\prod_{i=1}^n \frac{1}{A_i} = \Gamma\left(n\right)\int_{\mathbb{R}_+^n}\prod_{i=1}^n \frac{\delta\left(1-\sum_{i=1}^n y_i\right)}{\left(\sum_{i=1}^n A_iy_i\right)^n}\,dy_i. \tag{2.10}$$

Shapiro [17] states:

> An interesting anecdote of physics history is that Schwinger remained bitter that a virtually identical mathematical trick became commonly known as Feynman parameters. Why two brilliant physicists, each of whom had an appropriately won a Nobel prize, should fight over what is essentially a trivial mathematical trick, is an interesting question in the sociology of physicists.

An additional remark is that this representation is also known in probability theory, and was apparently introduced by Mauldon [13]. The n-dimensional Dirichlet distribution with parameters $\{\nu_i\}_{1\le i\le n}$ of a random vector X reads

$$f_X(x_1,\ldots,x_n) = \frac{\Gamma(\nu)}{\prod_{i=1}^n \Gamma(\nu_i)} \prod_{i=1}^n x_i^{\nu_i-1}\delta\left(1-\sum_{i=1}^n x_i\right)$$

with $\nu = \nu_1+\ldots+\nu_n$. Mauldon proved that a Dirichlet distributed random vector X satisfies the following equality

$$E_X\left(\sum_{i=1}^n \lambda_i X_i\right)^{-\nu} = \sum_{i=1}^n \lambda_i^{-\nu_i}, \tag{2.11}$$

which is exactly (2.4).

This procedure is now applied to the integral $N_{0,4}(a;m)$ written as

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(1+e^{i\theta}x^2)^{m+1}\,(1+e^{-i\theta}x^2)^{m+1}}\,dx \tag{2.12}$$

with $a = \cos\theta$ and $\theta \in (-\pi,\pi)$. The corresponding parameters are $n=2$ and $\nu_1=\nu_2=m+1$.

Recall the expression for the beta function

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \tag{2.13}$$

and its relation to the gamma function

$$B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \tag{2.14}$$

to write

$$N_{0,4}(a;m) = \frac{1}{B(m+1,m+1)}\int_0^\infty dx \int_0^1 \frac{x_1^m(1-x_1)^m\,dx_1}{\left[1+x^2\times(x_1e^{i\theta}+(1-x_1)e^{-i\theta})\right]^{2m+2}}.$$

Formula 3.194.3 in [9] gives

$$\int_0^\infty \frac{dx}{(1+ux^2)^\alpha} = \frac{1}{2\sqrt{u}}B\left(\tfrac12,\alpha-\tfrac12\right) \tag{2.15}$$

valid for $u\in\mathbb{C}$ with $|\mathrm{Arg}(u)|<\pi$. This yields

$$N_{0,4}(a;m) = \frac{B(\frac12,2m+\frac32)}{2B(m+1,m+1)}\int_0^1 \frac{x_1^m(1-x_1)^m\,dx_1}{\sqrt{x_1e^{i\theta}+(1-x_1)e^{-i\theta}}}. \tag{2.16}$$

The next lemma provides an evaluation of the integral in (2.16).

Lemma 2.1. Define

$$I_m(z) := \int_0^1 \frac{x^m(1-x)^m\,dx}{\sqrt{1+(z^2-1)x}}. \tag{2.17}$$

Then

$$I_m(z) = \frac{2^{2m+2}}{(m+1)\binom{2m+1}{m}\binom{4m+2}{2m+1}(z+1)^{2m+1}} \sum_{k=0}^{m} \binom{2m-2k}{m-k}\binom{m+k}{k}(z+1)^{2k}z^{m-k}.$$

Proof. Expand the integrand in powers of x and integrate. The result is simplified using the value for the beta function

$$B(a,b) = \frac{a+b}{ab}\binom{a+b}{a}^{-1} \tag{2.18}$$

for $a,\, b \in \mathbb{N}$. It follows that

$$\begin{aligned} I_m(z) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}(z^2-1)^k \int_0^1 x^{m+k}(1-x)^m\,dx \\ &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} B(m+k+1,m+1)(z^2-1)^k \\ &= \frac{1}{m+1}\sum_{k=0}^{\infty} \binom{2k}{k}\binom{2m+k+1}{m+1}^{-1}\left(\frac{1-z^2}{4}\right)^k. \end{aligned}$$

The result now follows from the identity

$$\sum_{k=0}^{\infty} \binom{2k}{k}\binom{2m+k+1}{m+1}^{-1}\left(\frac{1-z^2}{4}\right)^k = \frac{2^{2m+2}}{\binom{2m+1}{m}\binom{4m+2}{2m+1}(z+1)^{2m+1}} \sum_{k=0}^{m}\binom{2m-2k}{m-k}\binom{m+k}{k}(z+1)^{2k}z^{m-k}. \tag{2.19}$$

To establish (2.19), the Wilf-Zeilberger automated method [16] is employed. It provides the recurrence

$$(z^2-1)^2(4m+9)(4m+7)a_{m+2} - (z^4-6z^2+1)(2m+6)(2m+3)a_{m+1} - 4z^2(m+2)(m+3)a_m = 0$$

that both sides of (2.19) satisfy. Checking the initial values at $m=0$ and $m=1$ is elementary and the proof is complete. □

The evaluation of (1.2) is restated in a slightly different form, as the content of the next statement.

Theorem 2.2. Let $m \in \mathbb{N}$, $z = e^{i\theta}$ and $a = \cos\theta$. Then

$$N_{0,4}(a;m) = \frac{B(\frac{1}{2}, 2m+\frac{3}{2})}{2B(m+1,m+1)}\sqrt{z}I_m(z) \tag{2.20}$$

holds.

3. A connection with Schur functions

The integral

$$G_n(\mathbf{q}) = \frac{2}{\pi}\int_0^\infty \prod_{k=1}^{n} \frac{1}{x^2+q_k^2}\,dx, \tag{3.1}$$

is clearly a symmetric function of the parameters $q_1, \ldots, q_n$. It has been expressed in [2] as

$$G_n(\mathbf{q}) = \frac{s_{\lambda(n-1)}(\mathbf{q})}{e_n(\mathbf{q})\, s_{\lambda(n)}(\mathbf{q})}, \tag{3.2}$$

where $\lambda(n)$ is the partition $\lambda(n) = (n-1, n-2, \ldots, 1)$, $e_n(\mathbf{q}) = q_1q_2\cdots q_n$ and the Schur function corresponding to a partition μ defined by

$$s_\mu(\mathbf{q}) = \frac{a_{\mu+\lambda(n)}(\mathbf{q})}{a_{\lambda(n)}(\mathbf{q})} \tag{3.3}$$

with $a_\mu(\mathbf{q}) := \det(q_i^{\mu_j})_{1\leq i,j\leq n}$.

In the current problem, let $a = \frac{1}{2}(w^2+w^{-2})$ so that

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^2+w^2)^{m+1}\,(x^2+w^{-2})^{m+1}}. \tag{3.4}$$

Now take the partition λ as above and the $(2m+2)-$tuple $\mathbf{q}$ given by

$$\lambda(n) = (n-1,\, n-2, \ldots, 1), \quad \mathbf{q} = (w, \ldots, w, w^{-1}, \ldots, w^{-1}) \tag{3.5}$$

where $\mathbf{q}$ has $m+1$ copies of w and w^{-1}. The expression (3.2), that appears as Theorem 5.1 of [2], provides the next result.

Theorem 3.1. The quartic integral is given by

$$N_{0,4}(a;m) = \frac{\pi}{2}\frac{s_{\lambda(2m+1)}(\mathbf{q})}{s_{\lambda(2m+2)}(\mathbf{q})}. \tag{3.6}$$

As a corollary, this implies a specialization for the Schur functions.

Corollary 3.2. If $q = (w, \ldots, w, w^{-1}, \ldots, w^{-1})$ is of length $2m+2$, then

$$\frac{s_{\lambda(2m+1)}(w,\ldots,w,w^{-1},\ldots,w^{-1})}{s_{\lambda(2m+3)}(w,\ldots,w,w^{-1},\ldots,w^{-1})} = 2^{-2m}\sum_{k=0}^{m}\binom{2k}{k}\binom{2m-k}{m}\frac{1}{(w+w^{-1})^{2k+1}}.$$

Theorem 3.1 has a natural generalization to integrals of the form

$$\int_0^\infty \frac{dx}{Q_{2n}(x)^m},$$

where

$$Q_{2n}(x) = x^{2n} + a_1x^{2n-2} + a_2x^{2n-4} + \cdots + a_2x^4 + a_1x^2 + 1 \tag{3.7}$$

is a *palindromic polynomial*. These polynomials factor as

$$Q_{2n}(x) = (x^2+w_1^2)(x_2+w_1^{-2})\cdots(x^2+w_n^2)(x_2+w_n^{-2}). \tag{3.8}$$

Consider the partition λ and the n-concatenation $\mathbf{q} = (q_1, \ldots, q_n)$, of the $2m$-tuples q_j, by

$$\lambda(s) = (s-1, s-2, \ldots, 1), \text{ and } q_j = (w_j, \ldots, w_j; w_j^{-1}, \ldots, w_j^{-1}) \tag{3.9}$$

where q_j has m copies of each w_j and w_j^{-1}. A direct application of (3.2) gives the next result.

Proposition 3.3. Preserving the notation from above,

$$\int_0^\infty \frac{dx}{Q_{2n}(x)^m} = \frac{\pi}{2}\frac{s_{\lambda(2nm-1)}(q)}{s_{\lambda(2nm)(q)}}. \tag{3.10}$$

4. A connection with Bessel functions

The quartic integral (1.2) is evaluated in this section using the modified Bessel function $K_\alpha(x)$. The classical Bessel function $J_\alpha(x)$ is usually defined via the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1+\frac{\alpha^2}{x^2}\right)y = 0. \tag{4.1}$$

The modified function $I_\alpha(x) = e^{-\alpha\pi i/2}J_\alpha(xe^{\pi i/2})$ is also real valued. In the case $\alpha \notin \mathbb{Z}$, the functions $I_\alpha(x)$ and $I_{-\alpha}(x)$ form a basis for the solutions of (4.1). To deal with the situation of integer parameters, it is convenient to define

$$K_\alpha(x) = \frac{\pi}{2\,\sin\pi\alpha}\left[I_{-\alpha}(x) - I_\alpha(x)\right]. \tag{4.2}$$

This is the modified Bessel function of second kind, also called MacDonald function.

The proof of (1.2) proceeds along the following lines: the first step is to obtain an expression for $N_{0,4}(a,m)$ as an integral involving $K_{1/4}(x)$. An entry in [12] gives a hypergeometric form of this integral. The final step is to prove a hypergeometric identity that transforms this form to a result in [7]. Details about Bessel functions can be found in [4] and chapter 10 of [14].

The identity

$$\int_0^\infty e^{-cu}u^m\,du = \frac{\Gamma(m+1)}{c^{m+1}}, \tag{4.3}$$

valid for $\operatorname{Re} c > 0$, is used with $c = x^4+2ax^2+1$ to produce

$$\frac{1}{(x^4+2ax^2+1)^{m+1}} = \frac{1}{\Gamma(m+1)}\int_0^\infty e^{-u}u^m e^{-u(x^4+2ax^2)}\,du. \tag{4.4}$$

Integration yields

$$N_{0,4}(a;m) = \frac{1}{\Gamma(m+1)}\int_0^\infty e^{-u}u^m\int_0^\infty e^{-u(x^4+2ax^2)}\,du\,dx. \tag{4.5}$$

The restriction $\operatorname{Re} c > 0$ is satisfied by taking $-1 < a < 1$.

Entry 3.469.1 in [9] gives

$$\int_0^\infty e^{-\mu x^4-2\nu x^2}\,dx = \frac{1}{4}\sqrt{\frac{2\nu}{\mu}}\exp\left(\frac{\nu^2}{2\mu}\right)K_{1/4}\left(\frac{\nu^2}{2\mu}\right). \tag{4.6}$$

The choice $\mu = u$ and $\nu = au$ yields:

Lemma 4.1. The quartic integral $N_{0,4}(a;m)$ is given by

$$N_{0,4}(a;m) = \frac{2^{m-\frac{1}{2}}}{\Gamma(m+1)\,a^{2m+\frac{3}{2}}}\int_0^\infty t^m e^{-bt}K_{1/4}(t)\,dt, \tag{4.7}$$

where $b = 2/a^2-1$.

This expression can be evaluated using [12, vol.2, 2.16.6.2]

$$\int_0^\infty x^{\alpha-1}e^{-px}K_\nu\left(cx\right)dx = \frac{\left(2c\right)^\nu\sqrt{\pi}}{\left(p+c\right)^{\alpha+\nu}}\frac{\Gamma(\alpha-\nu)\,\Gamma(\alpha+\nu)}{\Gamma(\alpha+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}\alpha+\nu,\nu+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{matrix}\Big|\frac{p-c}{p+c}\right)$$

valid for $\mathrm{Re}\,(c+p)>0,\ \mathrm{Re}\,\alpha>|\,\mathrm{Re}\,\nu|$. This yields the expression

$$N_{0,4}(a;m) = \frac{\sqrt{\pi}a}{2\sqrt{2}m!}\frac{\Gamma\left(m+\frac{3}{4}\right)\Gamma(m+\frac{5}{4})}{\Gamma\left(m+\frac{3}{2}\right)}\,{}_2F_1\left(\begin{matrix}m+\frac{5}{4},\frac{3}{4}\\ m+\frac{3}{2}\end{matrix}\Big|1-a^2\right). \tag{4.8}$$

Using [1, 15.3.24] this can be written as

$$N_{0,4}\left(a;m\right) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(2m+\frac{3}{2}\right)}{\Gamma\left(2m+2\right)}\,{}_2F_1\left(\begin{matrix}2m+\frac{3}{2},\frac{1}{2}\\ m+\frac{3}{2}\end{matrix}\Big|\frac{1-a}{2}\right). \tag{4.9}$$

The proof of (1.2) is now based on the representation

$$N_{0,4}\left(a;m\right) = \frac{2^{m-\frac{1}{2}}}{(a+1)^{m+1/2}}B\left(2m+\tfrac{3}{2},\tfrac{1}{2}\right)\,{}_2F_1\left(\begin{matrix}-m,m+1\\ m+\frac{3}{2}\end{matrix}\Big|\frac{1-a}{2}\right) \tag{4.10}$$

described in [7]. In particular, the polynomial $P_m(a)$ is identified there as the Jacobi polynomial $P_m^{(m+1/2,-m-1/2)}(a)$.

Matching both representations for $N_{0,4}(a;m)$ shows that (1.2) follows from the next result.

Proposition 4.2. The identity

$${}_2F_1\left(\begin{matrix}-m,m+1\\ m+\frac{3}{2}\end{matrix}\Big|z\right) = (1-2z)(1-z)^{m+\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}m+\frac{5}{4},\frac{3}{4}\\ m+\frac{3}{2}\end{matrix}\Big|4z(1-z)\right) \tag{4.11}$$

holds.

Proof. The WZ-method shows that both sides satisfy the recurrence

$$\begin{aligned}(4m+7)(4m+9)za_{m+2}+(2m+3)(2m+5)(4z^2-4z-1)a_{m+1}&\\ -(2m+3)(2m+5)(z-1)a_m=0.&\end{aligned} \tag{4.12}$$

To complete the proof, it suffices to check two initial values. The required identities for $m=0$ and $m=1$ are written in terms of $t=4z(1-z)$ in the form

$${}_2F_1\left(\begin{matrix}\frac{5}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix}\Big|t\right) = \frac{\sqrt{2}}{(1-t)^{1/2}\sqrt{1+\sqrt{1-t}}} \tag{4.13}$$

and

$${}_2F_1\left(\begin{matrix}\frac{9}{4},\frac{3}{4}\\ \frac{5}{2}\end{matrix}\Big|t\right) = \frac{2\sqrt{2}(3+2\sqrt{1-t})}{5(1+\sqrt{1-t})^{3/2}\sqrt{1-t}}. \tag{4.14}$$

The proofs of these identities are elementary using the fact that the hypergeometric differential equation

$$t(1-t)y''-\left[c-(a+b+1)t\right]y'-aby=0. \tag{4.15}$$

has a unique solution analytic at $t=0$ with initial value $y(0)=1$. Indeed, both sides of (4.13) satisfy

$$t(1-t)y''+\left(\tfrac{3}{2}-3t\right)y'-\tfrac{15}{16}y=0 \tag{4.16}$$

with value $y(0)=1$. This establishes (4.13). The same holds for (4.14) using

$$t(1-t)y'' + \left(\tfrac{5}{2} - 4t\right) y' - \tfrac{27}{16} y = 0 \tag{4.17}$$

□

Note 4.3. The identity in Proposition 4.2 can also be established using C. Koutschan package `HolonomicFunctions` [11]. Denote the left hand side of (4.11) by $F1$ and the right hand side by $F2$. The command

$$\texttt{Annihilator}[F1, S[m]]$$

finds the operator

$$(7+4m)(9+4m)zS_m^2 + (3+2m)(5+2m)(-1-4z+4z^2)S_m - (3+2m)(5+2m)(-1+z)$$

with S_m being the shift in the discrete parameter m; that is, $S_m g(m;x) = g(m+1;x)$. The package claims that this operator annihilates the left hand side of (4.11). This is precisely the recurrence (4.12) obtained before. The command

$$\texttt{Annihilator}[F1, S[m]] == \texttt{Annihilator}[F2, S[m]]$$

returns `True`, showing that the right hand side $F2$ satisfies the same recurrence. The initial conditions can also be determined automatically.

5. A second proof of the main identity via Bessel functions

The previous section contains the identity (4.1) giving the quartic integral $N_{0,4}(a,m)$ as an integral involving Bessel functions:

$$N_{0,4}(a;m) = \frac{2^{m-\frac{1}{2}}}{\Gamma(m+1)\, a^{2m+\frac{3}{2}}} \int_0^\infty t^m e^{-bt} K_{1/4}(t)\, dt. \tag{5.1}$$

The table of integrals [9] contains, as Entry 6.611.3, the special case $m=0$:

$$\int_0^\infty e^{-bt} K_{1/4}(t)\, dt = \frac{\pi}{\sqrt{2}\sqrt{b^2-1}} \left[(b+\sqrt{b^2-1})^{1/4} - (b+\sqrt{b^2-1})^{-1/4} \right] \tag{5.2}$$

that can be written as

$$\int_0^\infty e^{-bt} K_{1/4}(t)\, dt = \frac{\pi}{2} \frac{a^{3/2}}{\sqrt{1+a}}, \tag{5.3}$$

using $b = a^2/2 - 1$. Differentiating m times with respect to b, it follows that

$$N_{0,4}(a,m) = \frac{(-1)^m 2^{m-1/2}}{m!\, a^{2m+3/2}} \frac{\partial^m}{\partial b^m} \left(\frac{\pi}{2} \frac{a^{3/2}}{\sqrt{1+a}} \right). \tag{5.4}$$

The proof of (1.2) is thus reduced to finding an analytic expression for the derivatives in (5.4).

Proposition 5.1. There exists a polynomial $Q_m(a)$ such that

$$\frac{\partial^m}{\partial b^m} \left(\frac{a^{3/2}}{\sqrt{1+a}} \right) = \frac{(-1)^m m!}{2^{2m}} \frac{a^{2m+3/2}}{(1+a)^{m+1/2}} Q_m(a). \tag{5.5}$$

Proof. The case $m=0$ holds with $Q_0(m)=1$. To complete the inductive step, it is shown that if $Q_m(a)$ is a polynomial then the function $Q_{m+1}(a)$ defined by the relation

$$\frac{\partial}{\partial b} \left(\frac{(-1)^m m!}{2^{2m}} \frac{a^{2m+3/2}}{(1+a)^{m+1/2}} Q_m(a) \right) = \frac{(-1)^{m+1}(m+1)!}{2^{2m+2}} \frac{a^{2m+7/2}}{(1+a)^{m+3/2}} Q_{m+1}(a), \tag{5.6}$$

is also a polynomial.

The chain rule and $\frac{\partial a}{\partial b} = -\frac{1}{4}a^3$ show that (5.6) is equivalent to

$$2(m+1)Q_{m+1}(a) = \left[2(m+1)(a+1) + (2m+1)\right]Q_m(a) + \left[2(a+1)^2 - 2(a+1)\right]Q'_m(a). \tag{5.7}$$

Thus $Q_{m+1}(a)$ is a polynomial in a. □

The proof of (1.2) is now reduced to checking that the polynomial Q_m in the previous lemma is given by $P_m(a)$ defined in (1.3).

Theorem 5.2. The polynomial $Q_m(a)$ is the same as $P_m(a)$.

Proof. It suffices to check that $P_m(a)$ satisfies the same recurrence as $Q_m(a)$. To this end, compare the coefficients of $(1+a)^k$ on both sides of (5.7). The result is equivalent to

$$\begin{aligned} 2^k(m+1)\binom{2m-2k+2}{m-k+1}\binom{m+k+1}{m+1} &= 2^{k+1}(m+1)\binom{2m-2k+2}{m-k+1}\binom{m+k-1}{m} \\ &+ 2^{k+1}(2m+1)\binom{2m-2k}{m-k}\binom{m+k}{m} \\ &+ 2^{k+1}(k-1)\binom{2m-2k+2}{m-k+1}\binom{m+k-1}{m} \\ &- 2^{k+2}k\binom{2m-2k}{m-k}\binom{m+k}{m}. \end{aligned}$$

This however can routinely be verified. Simply divide through by $\binom{2m-2k}{m-k}\binom{m+k}{m}$ and the statement reduces to a simple polynomial identity. □

A small variation of this proof of (1.2) is obtained by differentiating

$$\int_0^\infty e^{-bt}K_{\frac{1}{4}}(t)\,dt = \frac{\pi}{2^{1/4}(b+1)^{1/2}\sqrt{\sqrt{2}+\sqrt{b+1}}}. \tag{5.8}$$

directly with respect to the parameter b. The first few examples suggest the next result.

Lemma 5.3. There are polynomials S_m, T_m such that

$$\frac{d^m}{db^m}\left(\frac{1}{(b+1)^{1/2}\sqrt{\sqrt{2}+\sqrt{b+1}}}\right) = (-1)^m\frac{S_m(b)+\sqrt{b+1}T_m(b)}{2^{2m}(b+1)^{m+1/2}\left(\sqrt{2}+\sqrt{b+1}\right)^{m+1/2}}. \tag{5.9}$$

Proof. The proof is by induction on m, and is obtained upon differentiating the stated expression for the m-th derivative. The base case $m=0$ is obvious. Assume (5.9) holds for m. Then differentiating the right hand side of (5.9) generates the recurrence

$$\begin{aligned} S_{m+1}(b) &= 2\sqrt{2}(2m+1)S_m(b) \\ & \quad -(b+1)\left[4\sqrt{2}S_m'(b) - (1+6m)T_m(b) + 4(b+1)T_m'(b)\right] \\ T_{m+1}(b) &= 3(2m+1)S_m(b) + 4\sqrt{2}mT_m(b) \\ & \quad -4(b+1)\left(S_m'(b) + \sqrt{2}T_m'(b)\right), \end{aligned} \tag{5.10}$$

where $b = 2/a^2 - 1$. The initial condition $S_0(b) = 1$ and $T_0(b) = 0$ yield the result. $\square$

Lemma 5.4. Let $m' = \lfloor m/2 \rfloor$. Define

$$U_m(b) = \frac{2^{-2m}}{(1+b)^{m'}} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} 2^j (1+b)^{m'-j}$$

and

$$V_m(b) = \frac{2^{-2m}}{(1+b)^{m'}} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2j+1} 2^j (1+b)^{m'-j}.$$

Then

$$P_m(a) = U_m(b) + aV_m(b). \tag{5.11}$$

Proof. The polynomial $P_m(a)$ is decomposed into its even and odd part using

$$(1+a)^k = \frac{1}{2}\left[(1+a)^k + (1-a)^k\right] + \frac{1}{2}\left[(1+a)^k - (1-a)^k\right]. \tag{5.12}$$

The result follows. $\square$

The proof of (1.2) now reduces to expressing the polynomials $S_m(b)$ and $T_m(b)$ in terms of $U_m(b)$ and $V_m(b)$. This is given by

$$T_m(b) = \frac{m!2^{3m/2}}{\sqrt{2}a^{m-1}} \times \begin{cases} U_m(b) & \text{if } m \text{ is odd} \\ aV_m(b) & \text{if } m \text{ is even} \end{cases} \tag{5.13}$$

and

$$S_m(b) = \frac{m!2^{3m/2}}{\sqrt{2}a^{m}} \times \begin{cases} aV_m(b) & \text{if } m \text{ is odd} \\ U_m(b) & \text{if } m \text{ is even.} \end{cases} \tag{5.14}$$

The details are elementary and are left to the reader.

Note 5.5. The identity (5.4) yields

$$N_{0,4}(a,m) = \frac{\pi\, a^{m-1}}{m!2^{5m/2+3/2}\,(a+1)^{m+1/2}}\left[\sqrt{2}T_m(b) + S_m(b)\right]. \tag{5.15}$$

6. Conclusions

The value of the definite integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}},$$

is given by

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a)$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k.$$

Several proofs of this identity appeared in the literature. The current work includes proofs relating this identity to Bessel functions, Schur polynomials and a method of Schwinger for the evaluation of definite integrals.

Acknowledgements. The authors wish to thank Armin Straub for comments on C. Koutschan package. The work of the second author was partially supported by NSF-DMS 0070567. C. Vignat thanks V. Moll for his invitation to visit Tulane in July 2010.

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Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: tamdeber@tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu

Laboratoire des Signaux et Systemes, Universite d'Orsay, France
E-mail address: christophe.vignat@u-psud.fr