A SERIES INVOLVING CATALAN NUMBERS. PROOFS AND DEMONSTRATIONS.

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Abstract. An analytic expression for the generating function of the reciprocal of Catalan numbers is established by a variety of methods. These include some traditional proofs as well as one based on symbolic computations.

1. Introduction

In a recent issue of The American Mathematical Monthly, the readers will find **Problem 11765**. Proposed by David Beckwith, Sag Harbor, NY. Let C_n be the n-th Catalan number, defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$. Show that

$$\sum_{n=0}^{\infty} \frac{2^n}{C_n} = 5 + \frac{3\pi}{2} \text{ and } \sum_{n=0}^{\infty} \frac{3^n}{C_n} = 22 + 8\sqrt{3}\pi.$$

The question can be made slightly more challenging by asking:
Modified problem 11765. Find the values of the series

$$\sum_{n=0}^{\infty} \frac{2^n}{C_n} \text{ and } \sum_{n=0}^{\infty} \frac{3^n}{C_n}.$$

It is often the case that the solution of a problem becomes easier if one becomes more ambitious and aims to answer a more general question. In this context, we ask:
Generalized question. Find a closed-form formula for

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{C_n}. \tag{1.1}$$

The ratio test shows that the series converges for $|z| < 4$, this will be reflected in the expressions obtained for $f(z)$. The original questions correspond to the values $f(2)$ and $f(3)$. The reader will find a nice introduction to Catalan numbers in [9].

The results presented here come from our effort to produce different forms to solve this problem. The goal is to use a variety of methods that illustrate the approach that the authors use in the evaluation of series and integrals. The original question is relatively simple, so it is a perfect example to motivate these methods. The reader will travel to the world of Special Functions, Symbolic Computational Systems/Languages, Automatic Proofs and Probability. Many of the solutions will lead to the evaluation of definite integrals. Aside from classical and symbolic methods, the authors have chosen the table of integrals by I. S. Gradshteyn and

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I. M. Ryzhik [6] as the main source for definite integrals. The reliability of the entries in this table is of interest to the fourth author.

The word *Show* in Problem 11765 is interpreted here as *Prove*, so it is required to have a preliminary discussion on what constitutes a *mathematical proof.* A search in the literature produces a large variety of documents involving a discussion on this topic. For instance, some lecture notes for an `Introduction to mathematical arguments` by Michael Hutchings, begins with the following statement: *A mathematical **proof** is an argument which convinces other people that something is true.* A colloquial mathematical joke on this topic is: *'You only need to convince three people: one editor and two referees'.* Henry McKean [11, p. 104] provides a quote attributed to Mark Kac: *'A demonstration is to convince a reasonable man, a proof is to convince an unreasonable man'*. Under this point of view, most of the arguments presented here fall under the category of demonstrations. The reader will find in [2] and [17] some discussions on the role of computers in proofs, [7] and [8] describe the role of proof in Mathematics and Physics, [14] describes the role of proof in the progress in Mathematics and [5] presents an analysis on the role of proofs in the classroom.

2. The first proof: a `Mathematica` evaluation

In the second decade of the 21st century, it is natural to approach the question above as follows: *What does a Computer Algebra system say about the value of this series?* The authors use `Mathematica` and in version 9.0 simply input the line in boldface below to obtain the answer.

In[1]:= $\boldsymbol{Sum[2^n/CatalanNumber[n], \{n, 0, Infinity\}]}$ // ***Expand***
Out[1]= $5 + 3\pi/2$

The command *Expand* simply transforms the answer from $\frac{1}{2}(10+3\pi)$ to the form stated above. This gives the first evaluation. A similar `Mathematica` calculation produces the second one, this time the command *Function Expand* is used in the simplification. *Is this considered a mathematical proof?* There is a variety of sophisticated, well-tested algorithms behind the evaluation presented above. Given enough time, one could run the algorithm by hand and verify each of the steps. *Would that constitute a proof?*

3. The generalization

In the process of solving a question, such as the one proposed here, the authors always keep in mind how to explain their approach to students. The statement of the current problem leads naturally to the following question: *Is it possible to replace the values* 2 *and* 3 *by a general variable?* In other words, is it possible to produce a closed form for the generating function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{C_n}. \tag{3.1}$$

This is the generalized question mentioned in Section 1.

Using `Mathematica` again,

In[2]:= $\boldsymbol{Sum[z^n/CatalanNumber[n], \{n, 0, Infinity\}]}$
Out[2]= Hypergeometric $[1, 2, 1/2,\ z/4]$.

The answer now involves the *hypergeometric function*

$$ {}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{3.2} $$

with

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1) \tag{3.3} $$

the Pochhammer symbol. These might be expressed in terms of the gamma function

$$ \Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt, \qquad \text{for } \operatorname{Re} s > 0 \tag{3.4} $$

as

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}. \tag{3.5} $$

The reader will find in [3] extensive information about this special function.

To the non-expert, the use of a symbolic package to evaluate a series, has had a positive effect: it lead him/her to one of the most beautiful non-elementary functions. *For a teacher of Mathematics, this is great.* Naturally, this brings many questions, the most basic of which is whether the expression for the generating function (3.1) can be simplified.

A power series $h(z) = \sum_{n=0}^{\infty} u_n z^n$ is called *hypergeometric* if the ratio $\frac{u_{n+1}}{u_n}$ is a rational function of n. Most functions encountered in elementary courses are hypergeometric. The exponential $h(z) = e^z$ is one of them, since in this case $u_n = \frac{1}{n!}$ and $\frac{u_{n+1}}{u_n} = \frac{1}{n+1}$ is a rational function of n. Naturally, the geometric series, for which $u_n = 1$ is also hypergeometric function. The canonical notation for these functions comes from the factorization

$$ \frac{u_{n+1} z^{n+1}}{u_n z^n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)}{(n+b_1)(n+b_2)\cdots(n+b_q)} \frac{\alpha z}{n+1} \tag{3.6} $$

where $\{-a_j : 1 \leq j \leq p\}$ are the zeros of the numerator and $\{-b_j : 1 \leq j \leq q\}$ are the zeros of the denominator. The constant α comes from the leading coefficients in the factorization of the rational function. The convention is to always include the factor $n!$ in the form of the series and write it as $h(z) = \sum_{n=0}^{\infty} u_n \frac{z^n}{n!}$. This can be accomplished by adjusting the definition of u_n and it produces the term $n+1$ in the denominator of (3.6). The notation is

$$ h(z) = {}_pF_q\left(\left. \begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix} \right| \alpha z \right). \tag{3.7} $$

As a second example of the representation of an elementary function: the reader can check directly that

$$ \frac{\operatorname{ArcTan} z}{z} = {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} & 1 \\ & \frac{3}{2} \end{matrix} \right| -z^2 \right). \tag{3.8} $$

To get the power series expansion of ArcTan z, expand $1/(1+z^2)$ in a geometric series and integrate term by term.

The hypergeometric representation for the generating function $f(z)$ in (3.1) comes from identifying

$$u_n = \frac{n!}{C_n} \tag{3.9}$$

and the computation

$$\frac{u_{n+1}z^{n+1}}{u_n z^n} = \frac{(n+1)(n+2)}{n+\frac{1}{2}}\frac{z}{4} \tag{3.10}$$

from which one can read the zeros $a_1 = 1$, $a_2 = 2$, the poles $b_1 = \frac{1}{2}$ to obtain

$$f(z) = {}_2F_1\left(\left. \begin{matrix} 1, 2 \\ \frac{1}{2} \end{matrix} \right| \frac{z}{4} \right). \tag{3.11}$$

This confirms the evaluation given by `Mathematica`.

4. A simplification

Now that a hypergeometric expression for the generating function has been produced, it remains to explore the possibility of transforming it to simpler functions. This is, in general, a complicated process. One more time, we return to `Mathematica` for help.

The direct command produces the expected answer:

In[3]:= ***Hypergeometric2F1[1, 2, 1/2, z/4] // FunctionExpand // FullSimplify***

Out[3]= $\frac{2}{(z-4)^2}\left(z+8+\frac{12\sqrt{z}}{\sqrt{4-z}}\text{ArcCsc}\left[\frac{2}{\sqrt{z}}\right]\right).$

This can be expressed as

$$f(z) = \frac{2}{(z-4)^4}\left(z+8+\frac{12\sqrt{z}}{\sqrt{4-z}}\text{ArcSin}\left(\frac{\sqrt{z}}{2}\right)\right). \tag{4.1}$$

The singularity at $z=4$, coming from the radius of convergence of the series, can be seen in this formula.

The values stated in the problem now follow from this formula. An elementary proof of (4.1) is presented next.

The first part of the argument is to produce a recurrence for the coefficients $1/C_n$. This has already appeared in the computation of the ratio (3.10):

$$(n+2)C_{n+1} = 2(2n+1)C_n. \tag{4.2}$$

This yields

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{z^n}{C_n} = \sum_{n=0}^{\infty} \frac{2(2n+1)}{n+2}\frac{z^n}{C_{n+1}} \\ &= 4\sum_{n=0}^{\infty}\frac{z^n}{C_{n+1}} - 6\sum_{n=0}^{\infty}\frac{z^n}{(n+2)C_{n+1}}. \end{aligned}$$

The goal is to express the right-hand side in terms of $f(z)$. As a first step, multiply by z to obtain

$$zf(z) = 4\sum_{n=1}^{\infty}\frac{z^n}{C_n} - 6\sum_{n=1}^{\infty}\frac{z^n}{(n+1)C_n}$$

and multiplying by z one more time produces

$$z^2 f(z) - 4zf(z) + 4z = -6 \sum_{n=1}^{\infty} \frac{z^{n+1}}{(n+1)C_n}.$$

The second step is to eliminate the term $n+1$ in the denominator. This is accomplished by differentiation. It follows that

$$z(z-4)f'(z) + 2(z+1)f(z) = 2. \tag{4.3}$$

To solve this equation, multiply by the integrating factor $(4-z)^{5/2}z^{-1/2}$ to obtain

$$\frac{d}{dz}\left(f(z)(4-z)^{5/2}z^{-1/2} \right) = -2(4-z)^{3/2}z^{-3/2}. \tag{4.4}$$

A direct `Mathematica` evaluation (or the change of variables $z = u^2$ and $v = 2\sin\theta$) gives

$$-\int 2(4-z)^{3/2}z^{-3/2}\,dz = 16(4-z)^{1/2}z^{-1/2} + 2(4-z)^{1/2}z^{1/2} + 24\text{ArcSin}\left(\frac{\sqrt{z}}{2}\right).$$

Integrate the right-hand side of (4.4), checking that the implied constant of integration vanishes, to produce the following statement.

Theorem 4.1. The generating function of the reciprocal of Catalan numbers is given by

$$f(z) = \frac{2(z+8)}{(4-z)^2} + \frac{24\sqrt{z}}{(4-z)^{5/2}}\text{ArcSin}\left(\frac{\sqrt{z}}{2}\right). \tag{4.5}$$

Naturally, the length of a solution to a mathematical question depends on what is assumed by the authors. Readers of this journal, who are familiar with [10], will know the power series expansion

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{\binom{2n}{n}} = \frac{4}{4-z^2}\left[1 + \frac{z\,\text{ArcSin}(z/2)}{\sqrt{4-z^2}}\right]. \tag{4.6}$$

To obtain the generating function (4.5), replace z^2 by z, multiply by z and differentiate.

The evaluation of a series by a computer algebra system often produces answers in terms of non-elementary functions. In the present problem, the authors wish to report that a mistake in the typing of the series for $f(x)$ gave an indication of the nature of this function. Originally the fourth author used `Mathematica` and obtained the evaluation

$$\sum_{n=1}^{\infty} \frac{z^n}{C_n} = -\frac{z(z-10)}{(4-z)^2} + \frac{24\sqrt{z}}{(4-z)^{5/2}}\text{ArcSin}\left(\frac{\sqrt{z}}{2}\right).$$

The incorrect initial index lead us to simpler answers.

5. A direct evaluation of $f(z)$

The starting point of a proof can be crucial in determining its length. This section presents an evaluation of the series (3.1) based on an integral representation for the reciprocal of the central binomial coefficients

$$\frac{1}{\binom{2n}{n}} = \frac{n!\,n!}{(2n)!} = \frac{\Gamma^2(n+1)}{\Gamma(2n+1)}. \tag{5.1}$$

This proof also appears in [13].

The basic connection

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{5.2}$$

with the *beta function* defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \tag{5.3}$$

yields

$$\frac{1}{\binom{2n}{n}} = nB(n,n+1). \tag{5.4}$$

This identity appears in [4, Exercise 10.2.2, p. 193].

Summing (5.4) for $n = 1, 2, \cdots$ and interchanging the series with the integral yields

$$\sum_{n=1}^{\infty} \frac{z^n}{\binom{2n}{n}} = \sum_{n=1}^{\infty} nz^n \int_0^1 t^n(1-t)^n \frac{dt}{t} = \int_0^1 \frac{z(1-t)\,dt}{[1-zt(1-t)]^2}.$$

This last integral can be evaluated by partial fractions to produce (4.6). The generating function (3.1) is now obtained as before.

6. A connection with the error function

This section presents another evaluation of the generating function $f(z)$ in (1.1) based on the the *error function*

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt \tag{6.1}$$

encountered by the reader in basic Probability courses.

The story begins with a modification of the generating function for $f(z)$ defined by

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{C_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)!}{(2n)!} z^n. \tag{6.2}$$

The relation between $f(z)$ and $h(z)$ is part of a general result of power series described next.

The standard notation for the coefficient of z^n in the expansion the power series for a function $R(z)$ is denoted by $[z^n]R$. The next statement is of Laplace transform type and it is obtained by term by term integration of the series on the right-hand side.

Lemma 6.1. Assume

$$A(z) = \int_0^{\infty} e^{-t} B(zt)\,dt. \tag{6.3}$$

Then

$$[z^n]B = n! \times [z^n]A. \tag{6.4}$$

Now apply this lemma to the function $h(z)$ in (6.2) to obtain an expression for $f(z)$. The integral in (6.3) is evaluated first using a symbolic language. Indeed, `Mathematica` produces

$$h(z) = 1 + \frac{z}{4} + \frac{1}{8} e^{z/4} \sqrt{\pi\, z}(z+6)\, \mathrm{erf}\left(\frac{\sqrt{z}}{2} \right). \tag{6.5}$$

This can be verified using the expansion

$$\mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}} {}_1F_1 \left(\begin{matrix} \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| -z^2 \right). \tag{6.6}$$

Lemma 6.1 now gives

$$f(z) = \int_0^\infty e^{-t} \left(1 + \frac{zt}{4} + \frac{1}{8} e^{zt/4} \sqrt{\pi z t}\, (zt+6)\, \mathrm{erf}\left(\frac{\sqrt{zt}}{2} \right) \right) dt. \tag{6.7}$$

The first two terms can be integrated in elementary terms. Make the change of variables $u = \sqrt{zt}/2$ and replace $\mathrm{erf}(u)$ by $1 - \mathrm{erfc}(u)$, where erfc is the *complementary error function* and then introduce the notation

$$H_{n,m}(b) = \int_0^\infty x^n e^{-bx^2}\ [\mathrm{erfc}(x)]^m\ dx. \tag{6.8}$$

After the computation of some elementary integrals (6.7) becomes

$$f(z) = 1 + \frac{z}{4} + \frac{12\pi\sqrt{z}}{(4-z)^{5/2}} - \frac{4\sqrt{\pi}}{z} \left[2H_{4,1}(4/z - 1) + 3H_{2,1}(4/z-1) \right]. \tag{6.9}$$

The integrals $H_{n,m}(b)$ have been discussed in [1]. It turns out that $H_{n,m}(b)$ satisfies the recurrence

$$H_{n,m}(b) = \frac{n-1}{2b} H_{n-2,m}(b) - \frac{m}{b\sqrt{\pi}} H_{n-1,m-1}(b+1), \tag{6.10}$$

with initial conditions

$$H_{n,0}(b) = \frac{1}{2} b^{-(n+1)/2} \Gamma\left(\frac{n+1}{2} \right) \text{ and } H_{0,1}(b) = \frac{\mathrm{Arctan}\ \sqrt{b}}{\sqrt{\pi b}}. \tag{6.11}$$

The recurrence produces

$$H_{2,1}(b) = \frac{1}{2b\sqrt{\pi}} \left(\frac{\mathrm{Arctan}\ \sqrt{b}}{\sqrt{b}} - \frac{1}{b+1} \right) \tag{6.12}$$

and

$$H_{4,1}(b) = \frac{1}{b\sqrt{\pi}} \left(\frac{3\mathrm{Arctan}\ \sqrt{b}}{4\, b^{3/2}} - \frac{3}{4b(b+1)} - \frac{1}{2(b+1)^2} \right). \tag{6.13}$$

Actually (6.12) appears as entry 8.258.5 of [6] in the form

$$\int_0^\infty e^{-bx} \sqrt{x}\, \mathrm{erfc}\ \sqrt{x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{\mathrm{Arctan}\ \sqrt{b}}{b^{3/2}} - \frac{1}{b(1+b)} \right). \tag{6.14}$$

There is a total of five entries in Section 8.258. All of them can be evaluated in terms of the family $H_{n,m}(b)$.

In order to conclude with the evaluation of $f(z)$, replace the values of $H_{2,1}$ and $H_{4,1}$ in (6.9) to obtain

$$f(z) = \frac{2(6\pi\sqrt{z} + (z+8)\sqrt{4-z})}{(4-z)^{5/2}} - \frac{24\sqrt{z}}{(4-z)^{5/2}} \text{Arctan}\left(\frac{\sqrt{z}}{\sqrt{4-z}} \right). \tag{6.15}$$

This is equivalent to (4.5).

7. A probabilistic approach

This final section discusses a probabilistic approach to the evaluation of (3.1). This point of view has been used in [15] to prove some binomial identities. It turns out that this will produce a nice detour into the world of special functions. Some background is presented first.

The continuous random variables X considered here have a probability density function: this is a nonnegative function $f_X(x)$, such that

$$\Pr(X \leq x) = \int_{-\infty}^{x} f_X(y)\, dy. \tag{7.1}$$

In particular, f_X must have total mass 1. Thus, computations of probabilities or related quantities, such as moments, can often be reduced to the evaluation of integrals. For instance, the expectation of a measurable function h of the random variable X is computed as

$$\mathbb{E}h(X) = \int_{-\infty}^{\infty} h(y) f_X(y)\, dy. \tag{7.2}$$

The particular choice $h(X) = X^n$ produces the n-th moment $\mathbb{E}(X^n)$. In elementary courses, the reader has been exposed to normal random variables, written as $X \sim N(0,1)$, with density

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \text{for } x \in \mathbb{R}, \tag{7.3}$$

and to exponential random variables, with probability density function

$$f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0; \\ 0 & \text{otherwise,} \end{cases} \tag{7.4}$$

with $\lambda > 0$.

The arguments presented here use random variables with a gamma distribution of shape parameter $a > 0$, written as $X \sim \Gamma(a)$. These are defined by the density function

$$f_X(x;a) = \begin{cases} \frac{1}{\Gamma(a)} x^{a-1} e^{-x}, & \text{for } x \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

Here $\Gamma(s)$ is the classical gamma function, defined in (3.4). The exponential distribution is the special case of the gamma distribution with shape parameter $a = 1$.

The evaluation of the generating function for the reciprocal of Catalan numbers (3.1) is now obtained by this probabilistic approach.

Start with the expansion

$$A(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2n)!} = \cosh \sqrt{z} \tag{7.6}$$

and take Γ_1 and Γ_2 two independent Gamma random variables with shape parameters 1 and 2, respectively. The corresponding density functions are

$$f_{\Gamma_1}(x) = e^{-x} \text{ and } f_{\Gamma_2}(x) = xe^{-x}, \tag{7.7}$$

with moments $\mathbb{E}[\Gamma_1^n] = n!$ and $\mathbb{E}[\Gamma_2^n] = (n+1)!$. The independence of Γ_1 and Γ_2 now produces $\mathbb{E}\left[(z\Gamma_1\Gamma_2)^n)\right] = n!(n+1)!z^n$.

The random variable $\Gamma = \Gamma_1\Gamma_2$ has distribution

$$\begin{aligned} f_\Gamma(x) &= \int_0^\infty \frac{1}{x_1} f_{\Gamma_1}(x_1) f_{\Gamma_2}\left(\frac{x}{x_1}\right) dx_1 \\ &= x\int_0^\infty \frac{1}{x_1^2} e^{-(x_1+x/x_1)}\, dx_1 \\ &= 2\sqrt{x}K_1(2\sqrt{x}), \end{aligned}$$

where $K_\nu(z)$ is the Bessel function with integral representation

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-(t+z^2/4t)}\, dt}{t^{\nu+1}} \tag{7.8}$$

appearing as entry 8.432.6 in [6]. Now

$$f(z) = \mathbb{E}\sum_{n=0}^\infty \frac{(z\Gamma_1\Gamma_2)^n}{(2n)!} = \mathbb{E}\cosh\sqrt{z\Gamma_1\Gamma_2} \tag{7.9}$$

gives the generating function $f(z)$ as a formidable integral

$$f(z) = \int_0^\infty \cosh(\sqrt{zt}) \times 2\sqrt{t}K_1(2\sqrt{t})\, dt. \tag{7.10}$$

The simpler looking version

$$f(z) = \frac{1}{2}\int_0^\infty t^2 \cosh(\gamma t)\, K_1(t)\, dt, \text{ with } \gamma = \tfrac{1}{2}\sqrt{z}, \tag{7.11}$$

is obtained by the natural change of variables $\sqrt{t} \mapsto t$.

The remainder of this section is dedicated to its evaluation. First observe that `Mathematica` gives

$$\frac{1}{2}\int_0^\infty t^2\cosh(\gamma t)\, K_1(t)\, dt = \frac{2+\gamma^2}{2(1-\gamma^2)^2} + \frac{3\gamma}{2(1-\gamma^2)^{5/2}}\text{ArcSin}\ \gamma \tag{7.12}$$

and, with $\gamma = \sqrt{z}/2$, this produces (4.6). Symbolic languages do perform.

The authors now propose the following challenge: produce a proof of the formula (7.11) *using only the formulas appearing in the table of integrals* [6]. Aside from promoting this table, the restriction is meant to reflect the way the authors work: given an integral, the first step is to check if it appears in this table.

To begin with, the integral is a linear combination of entry 6.621.3:

$$\begin{aligned} &\int_0^\infty t^{\mu-1}e^{-\alpha t}K_\nu(\beta t)\, dt = \\ &\qquad \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+1}} \frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2})} {}_2F_1\left(\left.\begin{matrix}\mu+\nu, \nu+\frac{1}{2} \\ \mu+\frac{1}{2}\end{matrix}\right| \frac{\alpha-\beta}{\alpha+\beta}\right), \end{aligned} \tag{7.13}$$

that in the special case $\mu = 3$, $\nu = 1$ and $\beta = 1$ gives

$$\int_0^\infty t^2 e^{-\alpha t} K_1(t)\, dt = \frac{32}{5(\alpha+1)^4}\, {}_2F_1\left(\left. \begin{matrix} 4, \frac{3}{2} \\ \frac{7}{2} \end{matrix} \right| \frac{\alpha-1}{\alpha+1} \right). \tag{7.14}$$

The relation $\cosh u = \frac{1}{2}(e^u + e^{-u})$ and the expression (3.11) show that the evaluation of the generating function $f(z)$ is equivalent to the hypergeometric identity

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 1, 2 \\ \frac{1}{2} \end{matrix} \right| \frac{z}{4} \right) = \\ \frac{128}{5}\left[\frac{1}{(\sqrt{z}+2)^4}\, {}_2F_1\left(\left. \begin{matrix} 4, \frac{3}{2} \\ \frac{7}{2} \end{matrix} \right| \frac{\sqrt{z}-2}{\sqrt{z}+2} \right) + \frac{1}{(\sqrt{z}-2)^4}\, {}_2F_1\left(\left. \begin{matrix} 4, \frac{3}{2} \\ \frac{7}{2} \end{matrix} \right| \frac{\sqrt{z}+2}{\sqrt{z}-2} \right) \right]. \end{aligned} \tag{7.15}$$

The proof of this identity begins with the application of Pfaff's formula to the right-hand side. This is one of the most basic transformation rules for hypergeometric functions and it states

$${}_2F_1\left(\left. \begin{matrix} a, b \\ c \end{matrix} \right| x \right) = (1-x)^{-a}\, {}_2F_1\left(\left. \begin{matrix} a, c-b \\ c \end{matrix} \right| \frac{x}{x-1} \right). \tag{7.16}$$

It can be easily be deduced from the integral representation

$${}_2F_1\left(\left. \begin{matrix} a \;\; b \\ c \end{matrix} \right| z \right) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt, \tag{7.17}$$

appearing as entry 9.111 in [6]. In the current problem, this has the positive effect of canceling the fourth powers and converts (7.15) into

$${}_2F_1\left(\left. \begin{matrix} 1, 2 \\ \frac{1}{2} \end{matrix} \right| x \right) = \frac{1}{10}\left[{}_2F_1\left(\left. \begin{matrix} 4, 2 \\ \frac{7}{2} \end{matrix} \right| \frac{1-\sqrt{x}}{2} \right) + {}_2F_1\left(\left. \begin{matrix} 4, 2 \\ \frac{7}{2} \end{matrix} \right| \frac{1+\sqrt{x}}{2} \right) \right], \tag{7.18}$$

with $z = 4x$.

The proof presented here has been restricted to use only what can be found in the table [6]. Therefore it is natural to search there for hypergeometric identities that look like (7.18). There are not so many identities of this type in [6], but fortunately entries 9.136.1 and 9.136.2 give

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix} \right| \frac{1 \pm \sqrt{x}}{2} \right) = \\ \frac{\Gamma(a+b+\frac{1}{2})\sqrt{\pi}}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}\, {}_2F_1\left(\left. \begin{matrix} a, b \\ \frac{1}{2} \end{matrix} \right| x \right) \mp \frac{2\sqrt{\pi x}\; \Gamma(a+b+\frac{1}{2})}{\Gamma(a)\Gamma(b)}\; {}_2F_1\left(\left. \begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{3}{2} \end{matrix} \right| x \right). \end{aligned} \tag{7.19}$$

And this is precisely what is needed. To prove (7.18) simply add both cases of (7.19) with the special values $a = 2$ and $b = 1$. The proof of (7.19) is a direct consequence of a basic hypergeometric identity [3, Formula 3.1.12, page 128]:

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 2a \;\; 2b \\ a+b+\frac{1}{2} \end{matrix} \right| \frac{z+1}{2} \right) = & \frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}\, {}_2F_1\left(\left. \begin{matrix} a \;\; b \\ \frac{1}{2} \end{matrix} \right| z^2 \right) \\ & - z\frac{\Gamma(a+b+\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(a)\Gamma(b)}\, {}_2F_1\left(\left. \begin{matrix} a+\frac{1}{2} \;\; b+\frac{1}{2} \\ \frac{3}{2} \end{matrix} \right| z^2 \right). \end{aligned} \tag{7.20}$$

The proof of the formula for the generating function of $\{1/C_n\}$ is now complete.

This section has shown that the evaluation of $f(z)$, the generating function of the reciprocal of Catalan numbers, is equivalent to the identity (7.18). Having established this hypergeometric identity, the evaluation of $f(z)$ has been established. Now it is natural to a ask whether it is possible to prove (7.18) by expanding both sides as power series and comparing coefficients of equal powers. A simple calculation, left to the reader, shows that this is equivalent to

$$\frac{1}{C_n} = \sum_{k=2n}^{\infty} \frac{(4)_k (2)_k}{5\, \left(\frac{7}{2}\right)_k\, k!\, 2^{k-2n}} \binom{k}{2n}, \tag{7.21}$$

for every $n \in \mathbb{N}$. *This brings us back to the reciprocal of Catalan numbers*. This is an unexpected turn of events. To finish our discussion, a direct proof of (7.21) is presented next. The authors have been unable to find identities of this type in the literature.

Denote the summand in (7.21) by $f(n,k)$. In order to normalize the sum to start at 0, write $g(n,k) = f(n,k+2n)$. Then

$$\frac{g(n,k+1)}{g(n,k)} = \frac{(k+2n+2)(k+2n+4)}{(k+1)(2k+4n+7)}. \tag{7.22}$$

The representation (3.6) now gives

$$\sum_{k=2n}^{\infty} \frac{(4)_k (2)_k}{5\, \left(\frac{7}{2}\right)_k\, k!\, 2^{k-2n}} \binom{k}{2n} = g(n,0)\, {}_2F_1\left(\left. \begin{matrix} 2n+2 \quad 2n+4 \\ 2n+\frac{7}{2} \end{matrix} \right| \frac{1}{2} \right). \tag{7.23}$$

In order to complete the proof, one needs a formula for the hypergeometric function at argument $\frac{1}{2}$. Now Gauss comes to the rescue. The required evaluation is his *second summation formula* (see [3, p. 148], [12, formula 15.4.28], [16, Chapter XIV, Exercise 13])

$${}_2F_1\left(\left. \begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \right| \frac{1}{2} \right) = \frac{\Gamma(\frac{1}{2})\, \Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{a+1}{2}\right)\, \Gamma\left(\frac{b+1}{2}\right)}. \tag{7.24}$$

Replacing in (7.23) and some slight simplification using Euler's duplication formula $\Gamma\left(n+\frac{1}{2}\right) = \dfrac{(2n)!\,\sqrt{\pi}}{2^{2n}\, n!}$ yields the desired result (7.21). It is too bad that (7.24) does not appear in [6]. *It should definitely be included in the next edition.*

8. Conclusions

A variety of methods have been used to prove formulas for the generating function of the reciprocal of the Catalan numbers. These methods include traditional proofs, some modern proofs based on algorithms included in symbolic languages and also a proof based only on entries of a classical table of integrals. The authors hoped to illustrate the usual ways in which they approach a problem.

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