AN EXTENSION OF A CRITERION FOR UNIMODALITY

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ABSTRACT. We prove that if $P(x)$ is a polynomial with nonnegative nondecreasing coefficients and n is a positive integer, then $P(x+n)$ is unimodal. Applications and open problems are presented.

1. Introduction

A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *unimodal* if there exists an index $0 \leq m^* \leq m$, called the *mode* of the sequence, such that d_j increases up to $j = m^*$ and decreases from then on, that is, $d_0 \leq d_1 \leq \cdots \leq d_{m^*}$ and $d_{m^*} \geq d_{m^*+1} \geq \cdots \geq d_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Unimodal polynomials arise often in combinatorics, geometry and algebra. The reader is referred to [3] and [4] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal.

A sequence of positive real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *logarithmic concave* (or *log concave* for short) if $d_{j+1}d_{j-1} \leq d_j^2$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is log concave then it is unimodal [5]. A sufficient condition for log concavity of a polynomial is given by the location of its zeros: if all the zeros of a polynomial are real and negative, then it is log concave and therefore unimodal [5]. A simple criterion for unimodality was established in [1]: if a_j is a nondecreasing sequence of positive real numbers, then

$$P(x+1) = \sum_{j=0}^{m} a_j (x+1)^j \tag{1.1}$$

$$= \sum_{j=0}^{m} d_j(m) x^j \tag{1.2}$$

is unimodal. This criterion is reminiscent of Brenti's criterion for log concavity [3]. A sequence of real numbers is said to have *no internal zeros* if $d_i, d_k \neq 0$ and $i < j < k$ imply $d_j \neq 0$. Brenti's criterion states that if $P(x)$ is a log concave polynomial with nonnegative coefficients and with no internal zeros, then $P(x+1)$ is log concave.

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In this paper we first prove that under the same conditions of [1] the polynomial $P(x+n)$ is unimodal for any $n \in \mathbb{N}$, the set of positive integers. We also characterize the unimodal sequences $\{d_j\}$ that appear in [1] and discuss the behavior of the coefficients of $P(x+1)$ for a unimodal polynomial $P(x)$. Numerical evidence suggests that the unimodality result is true for n real and positive. This remains to be investigated.

2. The extension

In this section we prove an extension of the main result in [1]. We start by establishing an elementary inequality.

Lemma 2.1. Let $m,\, n \in \mathbb{N}$ and $m_* := \lfloor \frac{m}{n+1} \rfloor$. Then $(n+1)m_* \leq m \leq (n+1)m_* + n$.

Proof This follows directly from $\frac{m}{n+1} - 1 < m_* \leq \frac{m}{n+1}$.

Theorem 2.2. Let $0 \leq a_0 \leq a_1 \cdots \leq a_m$ be a sequence of real numbers and $n \in \mathbb{N}$, and consider the polynomial

$$P(x) \quad = \quad a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m. \tag{2.1}$$

Then the polynomial $P(x+n)$ is unimodal with mode $m_* = \lfloor \frac{m}{n+1} \rfloor$.

We now restate Theorem 2.2 in terms of the coefficients of P.

Theorem 2.3. Let $0 \leq a_0 \leq a_1 \cdots \leq a_m$ be a sequence of real numbers and $n \in \mathbb{N}$. Then the sequence

$$q_j := q_j(m,n) = \sum_{k=j}^{m} a_k \binom{k}{j} n^{k-j} \tag{2.2}$$

is unimodal with mode $m_* = \lfloor \frac{m}{n+1} \rfloor$.

Proof The coefficients $q_j(m)$ in (1.2) are given by

$$q_j(m) \quad = \quad \sum_{k=j}^{m} a_k \binom{k}{j} n^{k-j} \tag{2.3}$$

so that Theorem 2.3 follows from Theorem 2.2. Now

$$(i+1)(q_{i+1}(m) - q_i(m)) \quad \leq \quad \sum_{k=i}^{m} a_k \binom{k}{i} n^{k-i-1} \left[k - (n+1)i - n \right]. \tag{2.4}$$

Suppose $m_* \leq i \leq m-1$. Then

$$k - (n+1)i - n \leq m - (n+1)i - n \leq m - (n+1)m_* - n \leq 0, \tag{2.5}$$

where we have employed the Lemma in the last step. We conclude that every term in the sum (2.4) is nonpositive. Thus for $m_* \leq i \leq m-1$ we have $q_{i+1}(m) \leq q_i(m)$.

Now assume $0 \leq i \leq m_* - 1$. We show that $q_{i+1}(m) \geq q_i(m)$. Observe that in this case the sum (2.4) contains terms of both signs, so the positivity of the sum is not apriori clear. Consider

$$\begin{aligned}(i+1)\left(q_{i+1}(m)-q_i(m)\right) &= \sum_{k=(n+1)i+n+1}^{m} a_k \binom{k}{i} n^{k-i-1}\left[k-(n+1)i-n\right] \\ &\quad - \sum_{k=i}^{(n+1)i+n-1} a_k \binom{k}{i} n^{k-i-1}\left[-k+(n+1)i+n\right] \\ &:= T_2 - T_1. \end{aligned} \tag{2.6}$$

Observe that

$$\begin{aligned} T_1 &= \sum_{k=i}^{(n+1)i+n-1} a_k \binom{k}{i} n^{k-i-1}\left[-k+(n+1)i+n\right] \\ &\leq a_{(n+1)(i+1)} \sum_{k=i}^{(n+1)i+n-1} \binom{k}{i} n^{(n+1)i+n-1-i-1}\left[-k+(n+1)i+n\right] \\ &\leq a_{(n+1)(i+1)} n^{(i+1)n-2} \sum_{k=i}^{(n+1)i+n-1} \binom{k}{i}\left[-k+(n+1)i+n\right]. \end{aligned}$$

The monotonicity of the coefficients of P was used in the first step.

The last sum can be evaluated (e.g. symbolically) as

$$\sum_{k=i}^{(n+1)i+n-1} \binom{k}{i}\left[-k+(n+1)i+n\right] = \frac{(\,(n+1)i+n+1)!}{(i+2)!\,(ni+n-1)!},$$

so that

$$\begin{aligned} T_1 &\leq a_{(n+1)(i+1)} n^{(i+1)n} \times \frac{((n+1)i+n+1)!}{n^2(i+2)!(ni+n-1)!} \\ &\leq a_{(n+1)(i+1)} n^{(i+1)n} \times \frac{((n+1)i+n+1)!}{(ni+2n)(ni+n)\,i!\,(ni+n-1)!}. \end{aligned}$$

Now observe that

$$\frac{((n+1)i+n+1)!}{(ni+2n)(ni+n)\,i!\,(ni+n-1)!} \leq \binom{(n+1)(i+1)}{i}.$$

The inequality $T_1 \leq T_2$ now follows since the upper bound for T_1 established above is the first term in the sum defining T_2.

Corollary 2.4. Let $0 \leq a_0 \leq a_1 \cdots \leq a_m$ be a sequence of real numbers, $n \in \mathbb{N}$, and

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

Then $P(x+n)$ has decreasing coefficients for $n \geq m$.

Example 2.5. Let $2 < a_1 < \cdots < a_p$ and $r_1, \cdots, r_p$ be two sequences of positive integers. Then the sequence

$$q_j := \sum_{k=j}^{m} n^{k-j} \binom{a_1 m}{k^{r_1}} \binom{a_2 m}{k^{r_2}} \cdots \binom{a_p m}{k^{r_p}} \binom{k}{j}, \quad 0 \leq j \leq m$$

is unimodal.

3. The converse of the original criterion

The original criterion for unimodality states that if $P(x)$ has positive nondecreasing coefficients, then $P(x+1)$ is unimodal. In this section we discuss the following inverse question:

Given a unimodal sequence $\{d_j : 0 \leq j \leq m\}$, is there a polynomial $P(x) = a_0 + a_1x + \cdots + a_mx^m$ with nonnegative nondecreasing coefficients such that

$$P(x+1) = \sum_{j=0}^{m} d_j x^j \tag{3.1}$$

We begin by expressing the conditions on $\{a_j\}$ that guaranteed unimodality of $P(x+1)$ in terms of the coefficients $\{d_j\}$. Recall that

$$d_j = \sum_{k=j}^{m} a_k \binom{k}{j} \tag{3.2}$$

and

$$a_j = \sum_{k=j}^{m} (-1)^{k-j} d_k \binom{k}{j}. \tag{3.3}$$

Lemma 3.1. Let $0 \leq j \leq m$. Then

$$a_j \geq 0 \iff d_j \geq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j}. \tag{3.4}$$

Proof This follows directly from (3.3).

Lemma 3.2. Let $0 \leq j \leq m-1$. Then

$$a_j \leq a_{j+1} \iff d_j \leq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1}.$$

Proof This follows directly from the identity

$$a_{j+1} - a_j = \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1} - d_j.$$

We now combine the previous two lemmas to produce a criterion for unimodality.

Theorem 3.3. Let $Q(x) = d_0 + d_1x + \cdots + d_mx^m$ and assume the coefficients $\{d_j\}$ satisfy the inequalities

$$\sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j} \leq d_j \leq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1}. \tag{3.5}$$

Then $Q(x)$ is a unimodal polynomial for which $P(x) := Q(x-1)$ has positive and nondecreasing coefficients. Furthermore, for any $n \in \mathbb{N}$, $Q(x+n)$ is unimodal with mode $\lfloor \frac{m}{n+2} \rfloor$.

Proof The first part follows from the previous two lemmas. For the second part, Theorem 3.3 shows that $Q(x-1)$ has nonnegative, nondecreasing coefficients, so Theorem 2.2 yields the result.

Note. The inequality (3.5) is always consistent. The difference between the upper and lower bound is

$$\sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1} - \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j}$$

$$= \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j+1} = a_{j+1},$$

so the difference is always nonnegative.

Note. It would be interesting to describe the precise range of the map $(a_0, a_1, \cdots, a_m) \mapsto (d_0, d_1, \cdots, d_m)$. This map is linear, so the image of the set $0 \leq a_0 \leq \cdots \leq a_m$ is a polyhedral cone. In this paper we state one simple restriction on this image.

Proposition 3.4. Let $a_j \geq 0$. Then $d_j \geq d_{j+1}$ for $j \geq \lfloor m/2 \rfloor$.

Proof This follows directly from

$$\begin{aligned} d_j - d_{j+1} &= \sum_{k=j}^{m} a_k \binom{k}{j} - \sum_{k=j+1}^{m} a_k \binom{k}{j+1} \\ &= a_j + \sum_{k=j+1}^{m} a_k \frac{k!\,(2j+1-k)}{(j+1)!(k-j)!} \end{aligned}$$

since every term in the last sum is nonnegative.

4. A criterion for log concavity

Any nonnegative differentiable function f that satisfies $f(0) = f(m) = 0$ and $f''(x) \leq 0$ yields the unimodal sequence $\{f(j) : 0 \leq j \leq m\}$. The next theorem shows that these sequences are always log concave.

Proposition 4.1. Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a unimodal polynomial with mode n. Assume in addition that $c_{j+1} - 2c_j + c_{j-1} \leq 0$. Then $P(x)$ is log concave.

Proof Let $j < n$, so that $c_j \geq c_{j-1}$. The condition on c_j can be written as $c_j - c_{j-1} \geq c_{j+1} - c_j$, so that

$$c_j c_j - c_j c_{j-1} \geq c_{j+1} c_{j-1} - c_j c_{j-1},$$

and thus the log concavity condition holds. The case $j \geq n$ is similar.

5. The motivating example

The original criterion for unimodality in [1] was developed in our study of the coefficients $d_l(m)$ of the polynomial

$$P_m(a) \quad = \quad \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{5.1}$$

considered in [2]. These coefficients are given explicitly by

$$d_l(m) \quad = \quad 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \tag{5.2}$$

and we have conjectured that $\{d_l(m)\}_{l=0}^m$ forms a log concave sequence. Unfortunately Proposition 4.1 does not settle this question. For example, for $m = 15$ the sequence of signs in $d_{j+1}(15) - 2d_j(15) + d_{j-1}(15)$, for $1 \leq j \leq 14$, is

$$sign(15) = \{+1, +1, +1, +1, +1, -1, -1, -1, -1, +1, +1, +1, +1, +1\},$$

so the condition fails.

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