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The integrals in Gradshteyn and Ryzhik. Part 11: The incomplete beta function

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Abstract. The table of Gradshteyn and Rhyzik contains some integrals that can be expressed in terms of the incomplete beta function. We describe some elementary properties of this function and use them to check some formulas in the mentioned table.

1. Introduction

The table of integrals [**1**] contains a large variety of definite integrals that involve the *incomplete beta* function defined here by the integral

$$\beta(a) = \int_0^1 \frac{x^{a-1}\,dx}{1+x}. \tag{1.1}$$

The convergence of the integral requires $a > 0$. Nielsen, who used this function extensively, attributed it to Stirling [**3**], page 17. The table [**1**] prefers to introduce first the *digamma function*

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.2}$$

and define $\beta(x)$ by the identity

$$\beta(x) = \frac{1}{2}\left(\psi\left(\tfrac{x+1}{2}\right) - \psi\left(\tfrac{x}{2}\right)\right). \tag{1.3}$$

This definition appears as **8.370** and (1.1) appears as **3.222.1**. Here

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{1.4}$$

is the classical gamma function. Naturally, both starting points for β are equivalent, and Corollary 2.2 proves (1.3). The value

$$\gamma := -\psi(1) = -\Gamma'(1) \tag{1.5}$$

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is the well-known *Euler's constant.*

In this paper we will prove elementary properties of this function and use them to evaluate some definite integrals in [**1**].

2. Some elementary properties

The incomplete beta function admits a representation by series.

Proposition 2.1. Let $a \in \mathbb{R}^+$. Then

$$\beta(a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{a+k}. \tag{2.1}$$

PROOF. The result follows from the expansion of $1/(1+x)$ in (1.1) as a geometric series. □

Corollary 2.2. The incomplete beta function is given by

$$\beta(a) = \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right]. \tag{2.2}$$

This is **8.370** in [**1**].

PROOF. The expansion for the digamma function ψ

$$\psi(t) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{t+k} - \frac{1}{k+1}\right) \tag{2.3}$$

has been discussed in [**2**]. Then

$$\psi\left(\frac{a}{2}\right) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{2}{a+2k} - \frac{1}{k+1}\right) \tag{2.4}$$

and

$$\psi\left(\frac{a+1}{2}\right) = -\gamma - \sum_{k=0}^{\infty}\left(\frac{2}{a+2k+1} - \frac{1}{k+1}\right). \tag{2.5}$$

The identity (2.2) comes from adding these two expressions. □

These properties are now employed to prove some functional relations of the incomplete beta function. The proofs will employ the identities

$$\psi(x+1) = \frac{1}{x} + \psi(x) \tag{2.6}$$

$$\psi(x) - \psi(1-x) = -\pi\cot(\pi x) \tag{2.7}$$

$$\psi(x+\tfrac{1}{2}) - \psi(\tfrac{1}{2}-x) = \pi\tan(\pi x) \tag{2.8}$$

that were established in [**2**].

REMARK 2.1. Several of the evaluations presented here will employ the special values

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \tag{2.9}$$

that appears as **8.365.4**, and

$$\psi\left(\tfrac{1}{2} \pm n\right) = -\gamma + 2\left(\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right), \tag{2.10}$$

that appears as **8.366.3**.

Many of the formulas in Section **4.271** employ the values

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}, \tag{2.11}$$

that appear as **8.366.11** and also **8.366.12/13**:

$$\psi'(\tfrac{1}{2} \pm n) = \frac{\pi^2}{2} \mp 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}. \tag{2.12}$$

Higher order derivatives are given by

$$\begin{aligned} \psi^{(n)}(1) &= (-1)^{n+1} n! \zeta(n+1) \text{ and} \\ \psi^{(n)}(\tfrac{1}{2}) &= (-1)^{n+1} n! (2^{n+1}-1) \zeta(n+1). \end{aligned}$$

Proposition 2.3. The incomplete beta function satisfies

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{2.13}$$

$$\beta(1-x) = \frac{\pi}{\sin \pi x} - \beta(x), \tag{2.14}$$

$$\beta(x+1) = \frac{1}{x} - \frac{\pi}{\sin \pi x} + \beta(1-x). \tag{2.15}$$

PROOF. Using (2.2) we have

$$\begin{aligned} \beta(x+1) &= \frac{1}{2}\left[\psi\left(\frac{x+2}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right] = \frac{1}{2}\left[\psi\left(\frac{x}{2}+1\right) - \psi\left(\frac{x+1}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{2}{x} + \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right] \\ &= \frac{1}{x} - \beta(x). \end{aligned}$$

This establishes (2.13). To prove (2.14) we start with

$$\beta(x) + \beta(1-x) = \frac{1}{2}\left[\psi\left(\frac{1}{2}+\frac{x}{2}\right) - \psi\left(\frac{x}{2}\right) + \psi\left(1-\frac{x}{2}\right) - \psi\left(\frac{1}{2}-\frac{x}{2}\right)\right].$$

The formula (2.14) now follows from (2.7) and (2.8). □

3. Some elementary changes of variables

The class of integrals evaluated here are obtained from (1.1) by some elementary manipulations.

Example 3.1. The change $x = t^p$ in (1.1) yields

$$\beta(a) = p \int_0^1 \frac{t^{ap-1}\, dt}{1+t^p}. \tag{3.1}$$

Replace a by $\frac{a}{p}$ to obtain **3.241.1**:

$$\int_0^1 \frac{t^{a-1}\, dt}{1+t^p} = \frac{1}{p}\beta\left(\frac{a}{p}\right). \tag{3.2}$$

Example 3.2. The special case $p = 2$ in Example 3.1 gives

$$\beta(a) = 2 \int_0^1 \frac{t^{2a-1}\, dt}{1+t^2}. \tag{3.3}$$

Choose $a = \frac{b+1}{2}$, and relabel the variable of integration as x, to obtain **3.249.4**:

$$\int_0^1 \frac{x^b\, dx}{1+x^2} = \frac{1}{2}\beta\left(\frac{b+1}{2}\right). \tag{3.4}$$

Example 3.3. The evaluation of **3.251.7**:

$$\int_0^1 \frac{x^a\, dx}{(1+x^2)^2} = -\frac{1}{4} + \frac{a-1}{4}\beta\left(\frac{a-1}{2}\right) \tag{3.5}$$

comes from the change of variables $t = x^2$ and integration by parts. Indeed,

$$\begin{aligned} \int_0^1 \frac{x^a\, dx}{(1+x^2)^2} &= \frac{1}{2}\int_0^1 t^{(a-1)/2} \frac{d}{dt}\frac{1}{1+t}\, dt \\ &= -\frac{1}{4} + \frac{a-1}{4}\int_0^1 \frac{t^{(a-3)/2}\, dt}{1+t}\, dt, \end{aligned}$$

and (3.5) has been established.

Example 3.4. Formula **3.231.2** states that

$$\int_0^1 \frac{x^{p-1} + x^{-p}}{1+x}\, dx = \frac{\pi}{\sin \pi p}. \tag{3.6}$$

The integrals is recognized as $\beta(p)+\beta(1-p)$ and its value follows from (2.14). Similarly, **3.231.4** is

$$\int_0^1 \frac{x^p - x^{-p}}{1+x}\, dx = \frac{1}{p} - \frac{\pi}{\sin \pi p}. \tag{3.7}$$

The integral is now recognized as $\beta(1+p) - \beta(1-p)$, and the result follows from (2.15).

Example 3.5. The evaluation of **3.244.1**:

$$\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\,dx = \frac{\pi}{q}\text{cosec}\frac{p\pi}{q} \tag{3.8}$$

is

$$I = \frac{1}{q}\left(\beta(p/q)+\beta(1-p/q)\right) \tag{3.9}$$

according to (3.2). The result now follows from (2.14).

Example 3.6. The evaluation of **3.269.2**:

$$\int_0^1 x\,\frac{x^p-x^{-p}}{1+x^2}\,dx = \frac{1}{p}-\frac{\pi}{2\sin(\pi p/2)} \tag{3.10}$$

is obtained by the change of variables $t=x^2$, that produces

$$I = \frac{1}{2}\int_0^1 \frac{t^{p/2}-t^{-p/2}}{1+t}\,dt = \frac{1}{2}\left[\beta\left(\frac{p}{2}+1\right)-\beta\left(1-\frac{p}{2}\right)\right]. \tag{3.11}$$

The result now follows from (2.15).

4. Some exponential integrals

In this section we present some exponential integrals that may be evaluated in terms of the β-function.

Example 4.1. The change of variables $x=e^{-t}$ in (1.1) gives

$$\beta(a) = \int_0^\infty \frac{e^{-at}\,dt}{1+e^{-t}}. \tag{4.1}$$

This appears as **3.311.2** in [**1**].

Example 4.2. The evaluation of **3.311.13**:

$$\int_0^\infty \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\text{cosec}\left(\frac{\pi p}{p+q}\right) \tag{4.2}$$

is achieved by the change of variables $t=(p+q)x$ that produces

$$\begin{aligned} I &= \frac{1}{p+q}\int_0^\infty \frac{e^{-pt/(p+q)}}{1+e^{-t}}\,dt + \frac{1}{p+q}\int_0^\infty \frac{e^{-qt/(p+q)}}{1+e^{-t}}\,dt \\ &= \frac{1}{p+q}\left[\beta\left(\frac{p}{p+q}\right)+\beta\left(1-\frac{p}{p+q}\right)\right]. \end{aligned}$$

The result now comes from (2.15).

5. Some trigonometrical integrals

In this section we present the evaluation of some trigonometric integrals using the β-function.

Example 5.1. The change of variables $x = \tan^2 t$ in (1.1) gives

$$\beta(a) = 2\int_0^{\pi/4} \tan^{2a-1} t\, dt. \tag{5.1}$$

Introduce the new parameter $b = 2a-1$ to obtain **3.622.2**:

$$\int_0^{\pi/4} \tan^b t\, dt = \frac{1}{2}\beta\left(\frac{b+1}{2}\right). \tag{5.2}$$

Example 5.2. The change of variables $x = \tan t$ in (3.5) gives

$$\int_0^{\pi/4} \tan^a t\, \cos^2 t\, dt = -\frac{1}{4} + \frac{a-1}{4}\beta\left(\frac{a-1}{2}\right). \tag{5.3}$$

Now use (2.13) to obtain

$$\beta\left(\frac{a-1}{2}\right) = \frac{2}{a-1} - \beta\left(\frac{a+1}{2}\right), \tag{5.4}$$

that converts (5.3) to

$$\int_0^{\pi/4} \tan^a t\, \cos^2 t\, dt = \frac{1}{4} + \frac{1-a}{4}\beta\left(\frac{a+1}{2}\right). \tag{5.5}$$

This is the form in which **3.623.3** appears in [**1**]. Using this form and (5.2) we obtain **3.623.2**:

$$\int_0^{\pi/4} \tan^a t\, \sin^2 t\, dt = -\frac{1}{4} + \frac{1+a}{4}\beta\left(\frac{a+1}{2}\right). \tag{5.6}$$

Example 5.3. The evaluation of **3.624.1**:

$$\int_0^{\pi/4} \frac{\sin^p x\, dx}{\cos^{p+2} x} = \frac{1}{p+1} \tag{5.7}$$

can be done by writing the integral as

$$I = \int_0^{\pi/4} \tan^{p+2} x\, dx + \int_0^{\pi/4} \tan^p x\, dx. \tag{5.8}$$

These are evaluated using (5.2) to obtain

$$I = \frac{1}{2}\beta\left(\frac{p+3}{2}\right) + \frac{1}{2}\beta\left(\frac{p+1}{2}\right). \tag{5.9}$$

The rule (2.13) completes the proof.

Example 5.4. The integral **3.651.2**

$$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin x\cos x} = \frac{1}{3}\left(\beta\left(\frac{\mu+2}{2}\right)+\beta\left(\frac{\mu+1}{2}\right)\right) \tag{5.10}$$

can be established directly using the integral definition of β given in (1.1). Simply observe that dividing the numerator and denominator of the integrand by $\cos^2 x$ yields, after the change of variables $t=\tan x$, the identity

$$\begin{aligned}\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin x\cos x} &= \int_0^{\pi/4} \frac{\tan^\mu x}{(\sec^2 x-\tan x)}\frac{dx}{\cos^2 x} \\ &= \int_0^1 \frac{t^\mu\,dt}{t^2-t+1} \\ &= \int_0^1 \frac{t^{\mu+1}+t^\mu}{t^3+1}\,dt.\end{aligned}$$

The change of variables $t=s^{1/3}$ gives the result.

The evaluation of **3.651.1**

$$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1+\sin x\cos x} = \frac{1}{3}\left(\psi\left(\frac{\mu+2}{2}\right)-\psi\left(\frac{\mu+1}{2}\right)\right) \tag{5.11}$$

can be established along the same lines. This part employs the representation **8.361.7**:

$$\psi(z) = \int_0^1 \frac{x^{z-1}-1}{x-1}\,dx-\gamma \tag{5.12}$$

established in [**2**].

Example 5.5. The elementary identity

$$\frac{1}{1-\sin^2 x\cos^2 x} = \frac{1}{2}\left(\frac{1}{1+\sin x\cos x}+\frac{1}{1-\sin x\cos x}\right) \tag{5.13}$$

and the evaluations given in Examples 5.11 and 5.10 gives a proof of **3.656.1**:

$$\tfrac{1}{12}\left(-\psi\left(\tfrac{\mu+1}{6}\right)-\psi\left(\tfrac{\mu+2}{6}\right)+\psi\left(\tfrac{\mu+4}{6}\right)+\psi\left(\tfrac{\mu+5}{6}\right)+2\psi\left(\tfrac{\mu+2}{6}\right)-2\psi\left(\tfrac{\mu+1}{6}\right)\right). \tag{5.14}$$

Example 5.6. The final integral in this section is **3.635.1**:

$$\int_0^{\pi/4} \cos^{\mu-1}(2x)\,\tan x\,dx = \frac{1}{2}\beta(\mu). \tag{5.15}$$

This is easy: start with

$$\tan x = \frac{\sin x}{\cos x} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin 2x}{1+\cos 2x}, \tag{5.16}$$

and use the change of variables $t=\cos 2x$ to produce the result.

6. Some hyperbolic integrals

This section contains the evaluation of some hyperbolic integrals using the β-function.

Example 6.1. The integral (4.1) can be written as

$$\beta(a) = \int_0^\infty \frac{e^{t(1/2-a)}\,dt}{e^{t/2}+e^{-t/2}}, \tag{6.1}$$

and with $t = 2y$ and $b = 2a-1$, we obtain **3.541.6**:

$$\int_0^\infty \frac{e^{-by}\,dy}{\cosh y} = \beta\left(\frac{b+1}{2}\right). \tag{6.2}$$

Example 6.2. Integration by parts produces

$$\begin{aligned}\int_0^\infty \frac{e^{-ax}\,dx}{\cosh^2 x} &= 2\int_0^\infty e^{-ax}\frac{d}{dx}\frac{1}{1+e^{-2x}}\,dx \\ &= -1 + 2a\int_0^\infty \frac{e^{-ax}\,dx}{1+e^{-2x}}.\end{aligned}$$

The change of variables $t = 2x$ now gives the evaluation of **3.541.8**:

$$\int_0^\infty \frac{e^{-ax}\,dx}{\cosh^2 x} = a\beta\left(\frac{a}{2}\right) - 1. \tag{6.3}$$

Example 6.3. The change of variables $t = e^{-x}$ gives

$$\int_0^\infty e^{-ax}\tanh x\,dx = \int_0^1 \frac{t^{a-1}-t^a}{1+t^2}\,dt, \tag{6.4}$$

and with $s = t^2$ we get

$$\begin{aligned} I &= \frac{1}{2}\int_0^1 \frac{s^{a/2-1}-s^{(a-1)/2}}{1+s}\,ds \\ &= \frac{1}{2}\left[\beta\left(\frac{a}{2}\right) - \beta\left(\frac{a}{2}+1\right)\right].\end{aligned}$$

The transformation rule (2.13) gives the evaluation of **3.541.7**:

$$\int_0^\infty e^{-ax}\tanh x\,dx = \beta\left(\frac{a}{2}\right) - \frac{1}{a}. \tag{6.5}$$

7. Differentiation formulas

Example 7.1. Differentiating (1.1) with respect to the parameter a yields

$$\int_0^1 \frac{x^{a-1}\,\ln x}{1+x}\,dx = \beta'(a), \tag{7.1}$$

that appears as **4.251.3** in [**1**].

Example 7.2. Differentiating (3.2) n times with respect to the parameter a produces **4.271.16** written in the form

$$\int_0^1 \frac{x^{a-1}\,\ln^n x}{1+x^p}\,dx = \frac{1}{p^{n+1}}\beta^{(n)}\left(\frac{a}{p}\right). \tag{7.2}$$

The choice $n=1$ now gives formula **4.254.4** in [**1**]:

$$\int_0^1 \frac{x^{a-1}\,\ln x}{1+x^p}\,dx = \frac{1}{p^2}\beta'\left(\frac{a}{p}\right). \tag{7.3}$$

Example 7.3. The special case $n=1$, $a=1$ and $p=1$ in (7.2) produces the elementary integral **4.231.1**:

$$\int_0^1 \frac{\ln x\,dx}{1+x} = -\frac{\pi^2}{12}. \tag{7.4}$$

In this evaluation we have employed the values

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6}, \text{ and } \psi'(1/2) = \frac{\pi^2}{12}, \tag{7.5}$$

that appear in (2.12).

Example 7.4. Formula **4.231.14**:

$$\int_0^1 \frac{x\,\ln x}{1+x^2}\,dx = -\frac{\pi^2}{48} \tag{7.6}$$

comes from (7.3) by choosing the parameters $n=1$, $a=2$ and $p=2$. The values of $\psi'(1)$ and $\psi'(1/2)$ are employed again. Naturally, this evaluation also comes from (7.4) via the change of variables $x^2 \mapsto x$.

Example 7.5. The choice $n=a=1$ and $p=2$ in (7.3) and the values

$$\psi^{(2)}\left(\tfrac{1}{4}\right) = \pi^2 + 8G \text{ and } \psi^{(2)}\left(\tfrac{3}{4}\right) = \pi^2 - 8G, \tag{7.7}$$

where G is *Catalan constant* defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{7.8}$$

yields the evaluation of **4.231.12**:

$$\int_0^1 \frac{\ln x\,dx}{1+x^2} = -G. \tag{7.9}$$

The change of variables $x = t/a$, with $a>0$, and the elementary integral

$$\int_0^a \frac{dt}{t^2+a^2} = \frac{\pi}{4a}, \tag{7.10}$$

give the evaluation of **4.231.11**:

$$\int_0^a \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi\ln a - 4G}{4a}. \tag{7.11}$$

Example 7.6. Now choose $n = 1$, $a = 2$ and $p = 1$ in (7.3) and use the value $\psi'(3/2) = \pi^2/2 - 4$ given in (2.12) to obtain **4.231.19**:

$$\int_0^1 \frac{x \, \ln x}{1+x} \, dx = \frac{\pi^2}{12} - 1. \tag{7.12}$$

Combining this with (7.4) gives **4.231.20**:

$$\int_0^1 \frac{1-x}{1+x} \, \ln x \, dx = 1 - \frac{\pi^2}{6}. \tag{7.13}$$

Example 7.7. The values

$$\psi^{(2)}\left(\tfrac{1}{4}\right) = -2\pi^3 - 56\zeta(3) \text{ and } \psi^{(2)}\left(\tfrac{3}{4}\right) = 2\pi^3 - 56\zeta(3), \tag{7.14}$$

given in [**4**], are now used to produce the evaluation of **4.261.6**:

$$\int_0^1 \frac{\ln^2 x \, dx}{1+x^2} = \frac{\pi^3}{16}. \tag{7.15}$$

Example 7.8. The relation

$$\psi^{(n)}(1-z) + (-1)^{n+1}\psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot \pi z, \tag{7.16}$$

and the choice $n = 4$, $a = 1$ and $p = 2$ in (7.3) produces

$$\begin{aligned}
\int_0^1 \frac{\ln^4 x \, dx}{1+x^2} &= \frac{1}{2^5}\beta^{(4)}\left(\tfrac{1}{2}\right) \\
&= \frac{1}{1024}\left(\psi^{(4)}\left(\tfrac{3}{4}\right) - \psi^{(4)}\left(\tfrac{1}{4}\right)\right) \\
&= \frac{1}{1024}\left(-\pi \frac{d^4}{dz^4} \cot \pi z \Big|_{z=3/4}\right).
\end{aligned}$$

This yields the evaluation of **4.263.2**:

$$\int_0^1 \frac{\ln^4 x \, dx}{1+x^2} = \frac{5\pi^5}{64}. \tag{7.17}$$

The evaluation of **4.265**:

$$\int_0^1 \frac{\ln^6 x \, dx}{1+x^2} = \frac{61\pi^7}{256}, \tag{7.18}$$

can be checked by the same method.

Example 7.9. Now choose $n \in \mathbb{N}$ and take $a = n+1$ and $p = 1$ in (7.3) to obtain the expression

$$I := \int_0^1 \frac{x^n \, \ln^2 x}{1+x} \, dx = \frac{1}{8}\beta^{(2)}(n+1). \tag{7.19}$$

This is now expressed in terms of the $\psi-$function and then simplified employing the relation

$$\psi^{(m)}(z) = (-1)^{m+1} m! \zeta(m+1, z), \tag{7.20}$$

with the Hurwitz zeta function

$$\zeta(s,z) := \sum_{k=0}^{\infty} \frac{1}{(z+k)^s}. \tag{7.21}$$

We conclude that

$$I = \frac{1}{4}\left(\zeta\left(3, \frac{n+1}{2}\right) - \zeta\left(3, \frac{n+2}{2}\right)\right). \tag{7.22}$$

The elementary identity

$$\zeta\left(s, \tfrac{a}{2}\right) - \zeta\left(s, \tfrac{a+1}{2}\right) = 2^s \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s}, \tag{7.23}$$

is now used with $s=3$ and $a=n+1$ to obtain

$$\int_0^1 \frac{x^n \, \ln^2 x \, dx}{1+x} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n+1)^3}. \tag{7.24}$$

This is finally transformed to the form

$$\int_0^1 \frac{x^n \, \ln^2 x \, dx}{1+x} = (-1)^n \left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n} \frac{(-1)^k}{k^3} \right). \tag{7.25}$$

This is **4.261.11** of [**1**].

The same method produces **4.262.4**:

$$\int_0^1 \frac{x^n \, \ln^3 x \, dx}{1+x} = (-1)^{n+1} \left(\frac{7\pi^4}{120} - 6\sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right). \tag{7.26}$$

Example 7.10. The method of the previous example yields the value of **4.262.1**:

$$\int_0^1 \frac{\ln^3 x \, dx}{1+x} = -\frac{7\pi^4}{120}. \tag{7.27}$$

Here we use $\psi^{(3)}(1) = \pi^4/15$ and $\psi^{(3)}(1/2) = \pi^4$.

Similarly, $\psi^{(5)}(1) = 8\pi^6/63$ and $\psi^{(5)}(1/2) = 8\pi^6$ yields **4.264.1**:

$$\int_0^1 \frac{\ln^5 x \, dx}{1+x} = -\frac{31\pi^6}{252}, \tag{7.28}$$

and $\psi^{(7)}(1) = 8\pi^8/15$ and $\psi^{(7)}(1/2) = 136\pi^8$ yields **4.266.1**:

$$\int_0^1 \frac{\ln^7 x \, dx}{1+x} = -\frac{127\pi^8}{240}. \tag{7.29}$$

Example 7.11. A combination of the evaluations given above produces **4.261.2**:

$$\int_0^1 \frac{\ln^2 x\, dx}{1-x+x^2} = \frac{10\pi^3}{81\sqrt{3}}. \tag{7.30}$$

Indeed,

$$\begin{aligned}
\int_0^1 \frac{\ln^2 x\, dx}{1-x+x^2} &= \int_0^1 \frac{1+x}{1+x^3} \ln^2 x\, dx \\
&= \int_0^1 \frac{\ln^2 x\, dx}{1+x^3} + \int_0^1 \frac{x\, \ln^2 x\, dx}{1+x^3} \\
&= \frac{1}{27} \left(\beta^{(2)}(\tfrac{1}{3}) + \beta^{(2)}(\tfrac{2}{3}) \right) \\
&= \frac{1}{216} \left(\psi^{(2)}\left(\tfrac{2}{3}\right) - \psi^{(2)}\left(\tfrac{1}{3}\right) + \psi^{(2)}\left(\tfrac{5}{6}\right) - \psi^{(2)}\left(\tfrac{1}{6}\right) \right) \\
&= \frac{\pi}{216} \left(\frac{d^2}{dz^2} \cot \pi z \Big|_{z=1/3} + \frac{d^2}{dz^2} \cot \pi z \Big|_{z=1/6} \right) \\
&= \frac{\pi}{216} \left(\frac{8\pi^2}{3\sqrt{3}} + 8\sqrt{3}\pi^2 \right) = \frac{10\pi^3}{81\sqrt{3}}.
\end{aligned}$$

Example 7.12. Replace n by $2n$ in (7.2) and set $a = p = 1$ to produce

$$\begin{aligned}
\int_0^1 \frac{\ln^{2n} x\, dx}{1+x} &= \beta^{(2n)}(1) \\
&= \frac{1}{2^{2n+1}} \left(\psi^{(2n)}(1) - \psi^{(2n)}\left(\tfrac{1}{2}\right) \right) \\
&= \frac{2^{2n}-1}{2^{2n}} (2n)! \zeta(2n+1).
\end{aligned}$$

This appears as **4.271.1**.

Example 7.13. The change of variables $t = bx$ in (7.2) produces

$$\begin{aligned}
\int_0^b \frac{t^{a-1}\, \ln t}{b^p + t^p} &= \frac{b^{a-p}}{p^2} \beta'\left(\tfrac{a}{p}\right) + b^{1-a} \ln b \int_0^b \frac{t^{a-1}\, dt}{b^p + t^p} \\
&= \frac{b^{a-b}}{p^2} \beta'\left(\tfrac{a}{p}\right) + \ln b \frac{b^{a-p}}{p} \beta\left(\tfrac{a}{p}\right).
\end{aligned}$$

The last integral was evaluated using (3.2).

Differentiate this identity with respect to the parameter b to obtain

$$\begin{aligned}
\int_0^b \frac{t^{a-1}\, \ln t}{(b^p + t^p)^2}\, dt &= \frac{b^{a-2p} \ln b}{2p} + \frac{p-a}{p^3} b^{a-2p} \beta'\left(\tfrac{a}{p}\right) \\
&- \frac{b^{a-2p}}{p^2} (1 + (a-p) \ln b) \beta\left(\tfrac{a}{p}\right).
\end{aligned} \tag{7.31}$$

The special case $a = b = p = 1$ yields **4.231.6**:

$$\int_0^1 \frac{\ln x\, dx}{(1+x)^2} = -\beta(1) = -\ln 2. \tag{7.32}$$

Similarly, the choice $a = 2$, $b = 1$ and $p = 2$ yields **4.234.2**:

$$\int_0^1 \frac{x \ln x\, dx}{(1+x)^2} = -\frac{1}{4}\beta(1) = -\frac{\ln 2}{4}. \tag{7.33}$$

Example 7.14. In this last example of this section we present an evaluation of **4.234.1**:

$$\int_1^\infty \frac{\ln x\, dx}{(1+x^2)^2} = \frac{G}{2} - \frac{\pi}{8}, \tag{7.34}$$

using the methods developed here. We begin with the change of variables $x \mapsto 1/x$ to transform the problem to the interval $[0, 1]$. We have

$$\int_1^\infty \frac{\ln x\, dx}{(1+x^2)^2} = -\int_0^1 \frac{x^2 \ln x\, dx}{(1+x^2)^2}. \tag{7.35}$$

Now choose $a = 3$, $b = -1$ and $p = 2$ in (7.31) to obtain

$$\int_0^1 \frac{x^2 \ln x\, dx}{(1+x^2)^2} = -\frac{1}{8}\beta'\left(\frac{3}{2}\right) + \frac{1}{4}\beta\left(\frac{3}{2}\right). \tag{7.36}$$

The value of (7.34) now follows from

$$\begin{aligned}
\tfrac{1}{4}\beta\left(\tfrac{3}{2}\right) &= \tfrac{1}{8}\left(\psi\left(\tfrac{5}{4}\right) - \psi\left(\tfrac{3}{4}\right)\right) \\
&= \tfrac{1}{8}\left(4 + \psi\left(\tfrac{1}{4}\right) - \psi\left(\tfrac{3}{4}\right)\right) \\
&= \tfrac{1}{2} - \tfrac{\pi}{8},
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{8}\beta'\left(\tfrac{3}{2}\right) &= \tfrac{1}{32}\left(\psi'\left(\tfrac{5}{4}\right) - \psi'\left(\tfrac{3}{4}\right)\right) \\
&= \tfrac{1}{32}\left(\psi'\left(\tfrac{1}{4}\right) - \psi'\left(\tfrac{3}{4}\right) - 16\right) \\
&= \tfrac{1}{32}\left(\zeta(2, \tfrac{1}{4}) - \zeta(2, \tfrac{3}{4}) - 16\right) \\
&= \tfrac{G}{2} - \tfrac{1}{2}.
\end{aligned}$$

8. One last example

In this section we discuss the evaluation of **3.522.4**:

$$\int_0^\infty \frac{dx}{(b^2 + x^2)\cosh \pi x} = \frac{1}{b}\beta\left(b + \frac{1}{2}\right). \tag{8.1}$$

The technique illustrated here will be employed in a future publication to discuss many other evaluations.

To establish (8.1), introduce the function

$$h(b, y) := \int_0^\infty e^{-bt} \frac{\cos yt}{\cosh t}\, dt. \tag{8.2}$$

This function is harmonic and bounded for $\text{Re}\, b > 0$. Therefore it admits a Poisson representation

$$h(b,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(0,u) \frac{b}{b^2+(y-u)^2}\, du. \tag{8.3}$$

The value $h(0,u)$ is a well-known Fourier transform

$$h(0,u) = \int_0^\infty \frac{\cos yt}{\cosh t}\, dt = \frac{\pi}{2\cosh(\pi y/2)}, \tag{8.4}$$

that appears as **3.981.3** in [**1**]. Therefore we have

$$h(b,y) = \frac{b}{2}\int_{-\infty}^{\infty} \frac{du}{\cosh(\pi u/2)\,[b^2+(y-u)^2]}. \tag{8.5}$$

The special value $y=0$ and (6.2) give the result (after replacing b by $2b$ and u by $2u$).

Note 8.1. Formula (8.1) can also be obtained by a direct contour integration. Details will be provided in a future publication.

We conclude with an interpretation of (8.1) in terms of the sine Fourier transform of a function related to $\beta(x)$. The proof is a simple application of the elementary identity

$$\int_0^\infty e^{xt}\sin bt\, dt = \frac{b}{b^2+x^2}. \tag{8.6}$$

The details are left to the reader.

Theorem 8.2. Let

$$\mu(x) := \int_0^\infty \frac{e^{-xt}\, dt}{\cosh t} = \beta\left(\frac{x+1}{2}\right). \tag{8.7}$$

Then **3.522.4** in (8.1) is equivalent to the identity

$$\int_0^\infty \mu(t)\sin xt\, dt = \mu\left(\frac{2x}{\pi}\right). \tag{8.8}$$

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