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The integrals in Gradshteyn and Ryzhik. Part 14: An elementary evaluation of entry 3.411.5

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Abstract. An elementary proof of an entry in the table of integrals by Gradshteyn and Ryzhik is presented.

1. Introduction

The compilation by I. S. Gradshteyn and I. M. Ryzhik [**6**] contains about 600 pages of definite integrals. Some of them are quite elementary; for instance, 4.291.1

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12} \tag{1.1}$$

is obtained by expanding the integrand as a power series and using the value

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}. \tag{1.2}$$

The latter is reminiscent of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{1.3}$$

The reader will find in [**3**] many proofs of the classical evaluation (1.3).

Most entries in [**6**] appear quite formidable and their evaluation requires a variety of methods and ingenuity. Entry 4.229.7

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\sqrt{2\pi}\right) \tag{1.4}$$

illustrates this point. Vardi [**13**] describes a good amount of mathematics involved in evaluating (1.4). The integral is first interpreted in terms of the derivative of the

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L-function

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} - \cdots \tag{1.5}$$

as

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = -\frac{\pi\gamma}{4} + L'(1). \tag{1.6}$$

Here γ is *Euler's constant*. Then $L'(1)$ is computed in terms of the *gamma function*. This is an unexpected procedure.

Any treatise such as [**6**], containing large amount of information is bound to have some errors. Some of them are easy to fix. For instance, formula 3.511.8 in the sixth edition [**5**] reads

$$\int_0^\infty \frac{dx}{\cosh^2 x} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{1.7}$$

The source given for this integral is formula **BI**(98)(25) from the table by Bierens de Haan [**2**], where it appears as

$$\int_0^\infty \frac{1}{e^t + e^{-t}} \frac{dt}{\sqrt{t}} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{1.8}$$

The change of variable $t = \sqrt{x}$ yields a correct version of (1.7):

$$\int_0^\infty \frac{dx}{\cosh(x^2)} = \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}. \tag{1.9}$$

It is now clear what happened: a typo produced (1.7). In the latest edition of the integral table [**6**], the editors have decided to replace this entry with

$$\int_0^\infty \frac{dx}{\cosh^2 x} = 1. \tag{1.10}$$

The right-hand side of (1.7) has been corrected.

The advent of computer algebra packages has not made these tables obsolete. The latest version of `Mathematica` evaluates (1.10) directly, but it is unable to produce (1.9).

Most of the errors in [**6**] are of the type: some parameter has been mistyped, an exponent has been misplaced, parameters are mistaken to be identical (a common mishap is μ and u appearing in the same formula). Despite of this fact, it is a remarkable achievement for such an endevour. The accuracy of [**6**] comes from the effort of many generations, beginning with [**9**] and also including [**7, 11**].

A different type of error was found by one of the authors. It turns out that entry 3.248.5 of [**6**] is incorrect. To explain the reason for looking at any specific entry requires some background. The stated entry 3.248.5 involves the rational function

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)^2} \tag{1.11}$$

and the result says

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}. \tag{1.12}$$

The encounter begins with the evaluation of

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{1.13}$$

in the form

$$N_{0,4}(a;m) = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+1/2}}, \tag{1.14}$$

where $P_m(a)$ is a polynomial of degree m. The reader will find in [**1, 10**] details about (1.13) and properties of the coefficients of P_m. It is rather interesting that $N_{0,4}(a,m)$ appears in the expansion of the double square root function

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}N_{0,4}(a;k-1)c^k. \tag{1.15}$$

Browsing [**5**] on a leisure day, and with double square roots in our mind, formula 3.248.5 caught our attention. After many failed attempts at the proof, a simple numerical integration showed that (1.12) is incorrect. In spite of our inability to evaluate this integral, we have produced many equivalent versions. The reader is invited to verify that, if $\sigma(x,p) := \sqrt{x^4+2px^2+1}$ then the integral in (1.12) is $I(\frac{5}{3},1)$, where

$$I(p,q) = \int_0^\infty \frac{dx}{\sqrt{\sigma(x,p)}\sqrt{\sigma(x,q)}\sqrt{\sigma(x,p)+\sigma(x,q)}}. \tag{1.16}$$

The correct value of (1.12) *has eluded us.*

The reader is surely aware that often typos or errors could have profound consequences. In a letter to Larry Glasser, regarding (1.12), we mistyped the function $\varphi(x)$ of (1.11) as

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)}. \tag{1.17}$$

Larry, a consumated integrator, replied with $\sqrt{3}\left(\text{Tanh}^{-1}\sqrt{2\omega} - \frac{1}{\sqrt{2}}\text{Tanh}^{-1}\sqrt{\omega}\right)$ where $\omega = \left(\sqrt{7}-\sqrt{3}\right)/2\sqrt{7}$. Beautiful, but it does not help with (1.12). The editors of [**5**] have found an alternative to this quandry: the latest edition [**6**] has no entry 3.248.5.

Another example of errors in [**6**] has been discussed in the American Mathematical Journal by E. Talvila [**12**]. Several entries, starting with 3.851.1 [**5**]

$$\int_0^\infty x\sin(ax^2)\sin(2bx)\,dx = \frac{b}{2a}\sqrt{\frac{\pi}{2a}}\left[\cos\frac{b^2}{a} + \sin\frac{a^2}{b}\right] \tag{1.18}$$

are shown to be incorrect. This time, the errors are more dramatic: the integrals are divergent. These entries do not appear in the latest edition [**6**].

The website `http://www.math.tulane.edu/~vhm/Table.html` has the goal *to provide proofs and context to the entries in* [**6**]. The example chosen for the present article is taken from Section 3.411 consisting of 32 entries. The integrands are combinations of rational functions of powers and exponentials and the domain of integration is the whole real line or the half line $(0, \infty)$. There is a single exception: entry 3.411.5 states that

$$\int_0^{\ln 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12}. \tag{1.19}$$

The next section presents an elementary proof of (1.19).

2. A reduction argument

The expansion of the integrand in (1.19) as a geometric series yields

$$\frac{x}{1-e^{-x}} = x + \sum_{k=1}^{\infty} x e^{-kx}. \tag{2.1}$$

Term-by-term integration produces the following expressions

$$\int_0^a \frac{x\,dx}{1-e^{-x}}\,dx = \frac{1}{2}a^2 - \sum_{k=1}^{\infty} \frac{e^{-ak}}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} - a \sum_{k=1}^{\infty} \frac{e^{-ak}}{k}. \tag{2.2}$$

The complexity of these three series decreases as one moves from left to right. We now compute each term in (2.2), individually.

The third series. Integrating the geometric series $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ yields $\sum_{n\geqslant 1} \frac{x^n}{n} = \ln(1-x)$, which is valid for $|x| < 1$. Evaluating at $x = e^{-a}$ gives

$$\sum_{k=1}^{\infty} \frac{e^{-ak}}{k} = \ln(1-e^{-a}). \tag{2.3}$$

The second series. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.4}$$

plays a prominent role in the evaluation of the remaining 31 entries in Section 3.411. Indeed, the first of these

$$\int_0^{\infty} \frac{x^{\nu-1}\,dx}{e^{\mu x}-1} = \frac{1}{\mu^{\nu}}\Gamma(\nu)\zeta(\nu) \tag{2.5}$$

is the classical integral representation for $\zeta(\nu)$. It is becoming that the special value

$$\zeta(2) = \frac{\pi^2}{6} \tag{2.6}$$

appears as the second series in (2.2).

The first series. The second series in (2.2) is the only remaining part, we are alluding to the function $\sum_{k \geqslant 1} x^k/k^2$ evaluated at $x = e^{-a}$. This is the famous *polylogarithm* studied by Euler. See the introduction to [**8**] for a historical perspective. Aside from the series representation

$$\text{PolyLog}(2, x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \tag{2.7}$$

there is a natural integral expression

$$\text{PolyLog}(2, x) = -\int_0^x \frac{\ln(1-t)}{t}\, dt. \tag{2.8}$$

Therefore, the identity (2.2) reduces to

$$\int_0^a \frac{x\, dx}{1-e^{-x}} = \frac{1}{2}a^2 - \text{PolyLog}[2, e^{-a}] + \frac{\pi^2}{6} - a\ln(1-e^{-a}), \tag{2.9}$$

and entry 3.411.5 corresponds to the special value

$$\text{PolyLog}[2, \tfrac{1}{2}] = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2. \tag{2.10}$$

Equivalently,

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} = \frac{\pi^2}{6} - \ln^2 2. \tag{2.11}$$

Euler proved the functional equation

$$\text{PolyLog}[2, x] + \text{PolyLog}[2, 1-x] = \frac{\pi^2}{6} - \ln x \ln(1-x) \tag{2.12}$$

for the polylogarithm function. In particular, the case $x = \frac{1}{2}$ gives (2.11).

3. An elementary computation of the first series

A series for $\ln^2 2$ can be obtained by squaring $\ln 2 = -\sum_{n \geqslant 1} \frac{1}{n2^n}$ so that

$$\begin{aligned} \ln^2 2 &= \left(\sum_{n=1}^{\infty} \frac{1}{n2^n}\right) \times \left(\sum_{m=1}^{\infty} \frac{1}{m2^m}\right) = \sum_{n,m \geqslant 1} \frac{1}{nm2^{n+m}} \\ &= \sum_{r=1}^{\infty} \left(\sum_{m=1}^{r-1} \frac{1}{(r-m)m}\right) \frac{1}{2^r}. \end{aligned}$$

The partial fraction decomposition $\frac{1}{(r-m)m} = \frac{1}{r}\left(\frac{1}{m} + \frac{1}{r-m}\right)$ gives

$$\ln^2 2 = \sum_{r=1}^{\infty} \frac{H_{r-1}}{r2^{r-1}}, \tag{3.1}$$

where $H_{r-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{r-1}$ is the harmonic number. Therefore,

$$\begin{aligned} \ln^2 2 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} &= \sum_{r=1}^{\infty} \frac{H_{r-1}}{r2^{r-1}} + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}k^2} \\ &= \sum_{r=1}^{\infty} \frac{1}{r2^{r-1}} \left(H_{r-1} + \frac{1}{r} \right) \\ &= \sum_{r=1}^{\infty} \frac{H_r}{r2^{r-1}}. \end{aligned}$$

It remains to verify that this last series is in fact $\zeta(2)$.

The representation of the harmonic number as

$$H_r = \int_0^1 \frac{1-x^r}{1-x}\, dx \tag{3.2}$$

gives the desired step. Indeed, if c_r is a sequence of real numbers and α is fixed, then

$$\sum_{r=1}^{\infty} c_r H_r \alpha^r = \int_0^1 \frac{1}{1-x} \sum_{r=1}^{\infty} (1-x^r) c_r \alpha^r \, dx. \tag{3.3}$$

Thus the function

$$f(x) = \sum_{r=1}^{\infty} c_r x^r \tag{3.4}$$

appears in the integral representation

$$\sum_{r=1}^{\infty} c_r H_r \alpha^r = \int_0^1 \frac{f(\alpha) - f(\alpha x)}{1-x}\, dx. \tag{3.5}$$

In the present case, $c_r = 1/r2^{r-1}$ and $f(x) = -2\ln(1-x/2)$. Therefore,

$$\sum_{r=1}^{\infty} \frac{H_r}{r2^{r-1}} = \int_0^1 \frac{2\ln 2 + 2\ln(1-x/2)}{1-x}\, dx = 2\int_0^1 \frac{\ln(1+y)}{y}\, dy. \tag{3.6}$$

This last integral is computable via (1.1) and we have come full circle.

The technique described above in exhibiting an elementary proof of (2.11) can be applied to

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} = \zeta(3). \tag{3.7}$$

J. Borwein and D. Bradley [**4**] have given 32 proofs of this charming identity.

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