The integrals in Gradshteyn and Ryzhik.
Part 9: Combinations of logarithms, rational and trigonometric functions.

Tewodros Amdeberhan, Victor H. Moll, Jason Rosenberg, Armin Straub,
and Pat Whitworth

ABSTRACT. The table of Gradshteyn and Ryzhik contains many integrals with
integrands of the form $R_1(x) \, (\ln R_2(x))^m$, where R_1 and R_2 are rational functions.
In this paper we describe some examples where the logarithm appears to a single
power, that is $m = 1$, and the poles of R_1 are either real or purely imaginary.

1. Introduction

The table of integrals [**3**] contains many examples of the form

$$(1.1) \qquad \int_a^b R_1(x) \, (\ln R_2(x))^m \, dx,$$

where R_1 and R_2 are rational functions, a, $b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. For example, **4.231.1**
states that

$$(1.2) \qquad \int_0^1 \frac{\ln x \, dx}{1 + x} = -\frac{\pi^2}{12}.$$

This result can be established by the elementary methods described here.

Other examples, such as **4.233.1**

$$(1.3) \qquad \int_0^1 \frac{\ln x \, dx}{1 + x + x^2} = \frac{2}{9} \left(\frac{2\pi^2}{3} - \psi'(\tfrac{1}{3}) \right),$$

and **4.261.8**

$$(1.4) \qquad \int_0^1 \ln^2 x \, \frac{1 - x}{1 - x^6} \, dx = \frac{8\sqrt{3}\pi^3 + 351\zeta(3)}{486},$$

---

require more sophisticated special functions. Here ψ is the *digamma function* defined by

$$(1.5) \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

and $\zeta(s)$ is the classical *Riemann zeta function*. These results will be described in a future publication.

The integrals discussed here can also be framed in the context of trigonometric functions. For example, the change of variables $x = \tan t$ yields the identity

$$(1.6) \qquad \int_0^1 \frac{\ln x \, dx}{1 + x^2} = \int_0^{\pi/4} \ln \tan t \, dt = -G.$$

Here G is the *Catalan's constant* defined by

$$(1.7) \qquad G := \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

In this paper we concentrate on integrals of the type (1.1) where the logarithm appears to the first power and the poles of the rational function are either real of purely imaginary. The method of partial fractions and scaling of the independent variable, show that such integrals are linear combinations of

$$(1.8) \qquad h_{n,1}(b) := \int_0^b \frac{\ln t \, dt}{(1+t)^n},$$

and

$$(1.9) \qquad h_{n,2}(b) := \int_0^b \frac{\ln t \, dt}{(1+t^2)^n}.$$

The function $h_{n,1}$ was evaluated in [**4**], where is was denoted simply by h. We complete this evaluation in Section 4, by identifying a polynomial defined in [**4**]. The closed-form of $h_{n,1}$ involves the *Stirling numbers* of the first kind. The evaluation of $h_{n,2}$ is discussed in Section 6. The value of $h_{n,2}$ involves the *tangent integral*

$$(1.10) \qquad \text{Ti}_2(x) := \int_0^x \frac{\tan^{-1} t}{t}.$$

The case of integrals with more complicated pole structure will be described in a future publication.

2. Some elementary examples

We begin our discussion with an elementary example. Entry **4.291.1** states that

$$(2.1) \qquad \int_0^1 \frac{\ln(1+x)}{x} \, dx = \frac{\pi^2}{12}.$$

To establish this value we consider first a useful series. The result is expressed in terms of the *Riemann zeta function*

$$(2.2) \qquad \zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Lemma 2.1. Let $s > 1$. Then

$$(2.3) \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = -\frac{2^{s-1} - 1}{2^{s-1}} \zeta(s)$$

and

$$(2.4) \qquad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = \frac{2^s - 1}{2^s} \zeta(s)$$

PROOF. The second sum is

$$(2.5) \qquad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} - \sum_{k=1}^{\infty} \frac{1}{(2k)^s} = (1 - 2^{-s})\zeta(s).$$

To evaluate the first sum, split it into even and odd values of the index k:

$$(2.6) \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = \sum_{k=1}^{\infty} \frac{1}{(2k)^s} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s}$$

and use the value of the first sum. $\qquad \square$

To evaluate (2.1) we employ the expansion

$$(2.7) \qquad \ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

and integrate term by term we obtain

$$(2.8) \qquad \int_0^1 \frac{\ln(1+x)}{x}\,dx = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$

The result now follows from the lemma and the classical value $\zeta(2) = \pi^2/6$.

A similar calculation yields **4.291.2**:

$$(2.9) \qquad \int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}.$$

The change of variables $x = e^{-t}$ produces the evaluation of **4.223.1**

$$(2.10) \qquad \int_0^{\infty} \ln(1 + e^{-t})\,dt = \frac{\pi^2}{12}$$

and **4.223.2**:

$$(2.11) \qquad \int_0^{\infty} \ln(1 - e^{-t})\,dt = -\frac{\pi^2}{6}$$

3. More elementary examples

In [**4**] we analyze the case in which the rational function R_1 has a single multiple pole and $R_2(x) = x$. There are simple examples where the evaluation can be obtained directly. For instance, formula **4.231.1** states that

$$(3.1) \qquad \int_0^1 \frac{\ln x}{1 + x} \, dx = -\frac{\pi^2}{12}.$$

This can be established by simply expading the term $1/(1 + x)$ as a geometric series and integrate term by term. The same is true for **4.231.2**

$$(3.2) \qquad \int_0^1 \frac{\ln x}{1 - x} \, dx = -\frac{\pi^2}{6}.$$

The evaluation of **4.231.3**

$$(3.3) \qquad \int_0^1 \frac{x \ln x}{1 - x} \, dx = 1 - \frac{\pi^2}{6},$$

and **4.231.4**

$$(3.4) \qquad \int_0^1 \frac{1 + x}{1 - x} \ln x \, dx = 1 - \frac{\pi^2}{3},$$

follow directly from (3.2). Similar elementary algebraic manipulations produce **4.231.19**

$$(3.5) \qquad \int_0^1 \frac{x \ln x}{1 + x} \, dx = -1 + \frac{\pi^2}{2},$$

and **4.231.20**

$$(3.6) \qquad \int_0^1 \frac{(1 - x) \ln x}{1 + x} \, dx = 1 - \frac{\pi^2}{6}.$$

The same is true for **4.231.14**

$$(3.7) \qquad \int_0^1 \frac{x \ln x}{1 + x^2} \, dx = -\frac{\pi^2}{48},$$

and **4.231.15**

$$(3.8) \qquad \int_0^1 \frac{x \ln x}{1 - x^2} \, dx = -\frac{\pi^2}{24},$$

via the change of variables $t = x^2$.

The evaluation of **4.231.13**:

$$(3.9) \qquad \int_0^1 \frac{\ln x \, dx}{1 - x^2} = -\frac{\pi^2}{8}$$

is a direct consequence of the partial fraction decomposition

$$(3.10) \qquad \frac{1}{1 - x^2} = \frac{1}{2} \frac{1}{1 - x} + \frac{1}{2} \frac{1}{1 + x}$$

and the identites (3.1) and (3.2).

It is often the case that a simple change of variables reduces an integral to one that has previously been evaluated. For example, the change of variables $t = 1 - x^2$ produces

$$(3.11) \qquad \int_0^1 \frac{\ln(1 - x^2)}{x}\, dx = \frac{1}{2} \int_0^1 \frac{\ln t\, dt}{1 - t}.$$

The right-hand side is given in (3.2) and we obtain the value of **4.295.11**:

$$(3.12) \qquad \int_0^1 \frac{\ln(1 - x^2)}{x}\, dx = -\frac{\pi^2}{12}.$$

4. A single multiple pole

The situation for a single multiple pole is more delicate. The pole pole may be placed at $x = -1$ by scaling. The main result established in [**4**] is:

Theorem 4.1. Let $n \geqslant 2$ and $b > 0$. Define

$$(4.1) \qquad h_{n,1}(b) = \int_0^b \frac{\ln t\, dt}{(1 + t)^n}$$

and introduce the function

$$(4.2) \qquad q_n(b) = (1 + b)^{n-1} h_{n,1}(b).$$

Then

$$(4.3) \qquad q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1 + b) + Z_n(b),$$

where

$$(4.4) \qquad X_n(b) = \frac{(1 + b)^{n-1} - 1}{n - 1}, \quad Y_n(b) = -\frac{(1 + b)^{n-1}}{n - 1}.$$

Finally, the function

$$(4.5) \qquad T_n(b) := -\frac{(n - 1)!\, Z_n(b)}{b(1 + b)},$$

satisfies $T_2(b) = 0$ and for $n \geqslant 1$ it satisfies the recurrence

$$(4.6) \qquad T_{n+2}(b) = n(1 + b)T_{n+1}(b) + (n - 1)! \left(\frac{(1 + b)^n - 1}{b} \right).$$

It follows that $T_n(b)$ is a polynomial in b of degree $n-3$ with positive integer coefficients.

Note 4.2. The case $n = 1$ is expressed in terms of the *polylogarithm function*

$$(4.7) \qquad \mathrm{PolyLog}[n, z] := \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Indeed, we have

$$(4.8) \qquad \int_0^b \frac{\ln x}{1 + x}\, dx = \ln b \ln(1 + b) + \mathrm{PolyLog}[2, -b].$$

We now identify the polynomial T_n. The first few are given by

$$(4.9) \qquad \begin{aligned} T_3(b) &= 1, \\ T_4(b) &= 3b + 4, \\ T_5(b) &= 11b^2 + 27b + 18, \\ T_6(b) &= 50b^3 + 176b^2 + 216b + 96. \end{aligned}$$

For $n \geqslant 2$, define

$$(4.10) \qquad A_n(b) = \frac{1}{(n-2)!} T_n(b).$$

Then (4.6) becomes

$$(4.11) \qquad A_{n+2}(b) = (1+b)A_{n+1}(b) + \frac{(1+b)^n - 1}{bn},$$

with initial condition $A_2(b) = 0$.

The polynomial A_n is written as

$$(4.12) \qquad A_n(b) = \sum_{j=0}^{n-3} a_{n,j} b^j$$

The recursion (4.11) produces

Lemma 4.3. Let $n \geqslant 4$. Then the coefficients $a_{n,j}$ satisfy

$$(4.13) \quad \begin{aligned} a_{n,0} &= a_{n-1,0} + 1, \\ a_{n,j} &= a_{n-1,j} + a_{n-1,j-1} + \frac{(n-3)!}{(j+1)!\,(n-3-j)!}, \quad \text{for } 1 \leqslant j \leqslant n-4, \\ a_{n,n-3} &= a_{n-1,n-4} + \frac{1}{n-2}. \end{aligned}$$

The expressions $a_{n,0} = n - 2$ and $a_{n,n-3} = H_{n-2}$, the harmonic number, are easy to determine from (4.13). We now find closed-form expressions for the remaining coefficients. These involve the *Stirling numbers of the first kind* $s(n,j)$ defined by the expansion

$$(4.14) \qquad (x)_n = \sum_{j=1}^{n} s(n,j) x^j,$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol (also called rising factorial). The numbers $s(n,1)$ are given by

$$(4.15) \qquad s(n,1) = (-1)^{n-1}(n-1)!,$$

and the sequence $s(n,j)$ satisfies the recurrence

$$(4.16) \qquad s(n+1,j) = s(n,j-1) - ns(n,j).$$

Theorem 4.4. The coefficients $a_{n,j}$ are given by

$$(4.17) \qquad a_{n,j} = \frac{(-1)^j}{(j+1)!} \binom{n-2}{j+1} s(j+2,2).$$

PROOF. Define

$$(4.18) \qquad b_{n,j} := (-1)^j a_{n,j} \times (j+1)! \binom{n-2}{j+1}^{-1}.$$

The recurrence (4.13) shows that $b_{n,j}$ is independent of n and satisfies

$$(4.19) \qquad b_{n,j+1} = -j b_{n,j} + (-1)^{j+1} j!$$

and this is (4.16) in the special case $j = 2$. Formula (4.17) has been established. $\square$

We now restate the value of $h_{n,1}$. The index n is increased by 1 and the identity $|s(n,k)| = (-1)^{n+k} s(n,k)$ is used in order to make the formula look cleaner.

Corollary 4.5. Assume $b > 0$ and $n \in \mathbb{R}$. Then

$$(4.20) \qquad \int_0^b \frac{\ln t \, dt}{(1+t)^{n+1}} = \frac{1}{n}\left[1 - (1+b)^{-n}\right] \ln b - \frac{1}{n}\ln(1+b)$$

$$- \frac{1}{n(1+b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!}\binom{n-1}{j}|s(j+1,2)|b^j.$$

The special case $b = 1$ provides the evaluation

$$\int_0^1 \frac{\ln t \, dt}{(1+t)^{n+1}} = -\frac{\ln 2}{n} - \frac{1}{n 2^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!}\binom{n-1}{j}|s(j+1,2)|.$$

Elementary changes of variables, starting with $t = \tan^2 \varphi$, convert (4.20) into

$$(4.21) \quad \int_a^1 s^{n-1} \ln(1-s)\, ds = \frac{(1-a^n)}{n^2}\left[n\ln(1-a) - 1\right]$$

$$- \frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{j!}\binom{n-1}{j}|s(j+1,2)|a^{n+1-j}(1-a)^j.$$

The special case $a = 1/2$ produces

$$\int_{1/2}^1 s^{n-1} \ln(1-s)\, ds = \frac{1}{n 2^{n-1}} \sum_{j=1}^{n-1} \frac{\binom{n-1}{j}|s(j+1,2)|}{j!} - \frac{(2^n-1)}{n^2 2^n}(n\ln 2 + 1),$$

and $a = 0$ gives

$$(4.22) \qquad \int_0^1 s^{n-1}\ln(1-s)\, ds = -\left(\frac{1}{n^2} + \frac{|s(n,2)|}{n!}\right).$$

5. Denominators with complex roots

In this section we consider the simplest example of the type (1.1), where the rational function $R_1(x)$ has a complex (non-real) pole. This is

$$G := -\int_0^1 \frac{\ln x}{1 + x^2}\, dx.$$

The constant G is called *Catalan's constant* and is given by

$$(5.1) \qquad G = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}.$$

Entry **4.231.12** of [**3**] states

$$(5.2) \qquad \int_0^1 \frac{\ln x}{1 + x^2}\, dx = -G.$$

To verify (5.2) simply expand the integrand to produce

$$
\begin{aligned}
\int_0^1 \frac{\ln x\, dx}{1 + x^2} &= -\sum_{k=0}^\infty (-1)^k \int_0^1 x^{2k} \ln x\, dx \\
&= -\sum_{k=0}^\infty (-1)^k \int_0^\infty t e^{-(2k+1)t}\, dt \\
&= -\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} \int_0^\infty \sigma e^{-\sigma}\, d\sigma
\end{aligned}
$$

The integral is evaluated by integration by parts or recognizing its value as $\Gamma(2) = 1$.

Integration by parts gives the alternative form

$$(5.3) \qquad \int_0^1 \frac{\tan^{-1} x}{x}\, dx = G,$$

that appears as **4.531.1**.

There are many definite integrals in [**3**] that are related to Catalan's constant. For example, the change of variables $t = 1/x$ yields from (5.2), the value

$$(5.4) \qquad \int_1^\infty \frac{\ln t\, dt}{1 + t^2} = G.$$

This is the second part of 4.231.12. Adding these two expressions for G, we conclude that

$$(5.5) \qquad \int_0^\infty \frac{\ln x\, dx}{1 + x^2} = 0.$$

The change of variables $x = at$ in (5.2) yields **4.231.11**:

$$(5.6) \qquad \int_0^a \frac{\ln\, dx}{x^2 + a^2} = \frac{\pi \ln a}{4a} - \frac{G}{a}$$

We now introduce material that will provide a genaralization of (5.5) to the case of a multiple pole at i. The integral is expressed in terms of the *polygamma function*

$$(5.7) \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Lemma 5.1. Let a, $b \in \mathbb{R}$. Then

$$(5.8) \qquad \int_0^\infty \frac{\ln t \, dt}{(1+t^2)^b} = \frac{\Gamma(\frac{1}{2})\psi(\frac{1}{2}) - \Gamma(b - \frac{1}{2})\psi(b - \frac{1}{2})}{2\Gamma(b)}.$$

PROOF. Define

$$(5.9) \qquad f(a,b) := \int_0^\infty \frac{t^a \, dt}{(1+t^2)^b}.$$

Then

$$(5.10) \qquad \frac{d}{da} f(a,b) = \int_0^\infty \frac{t^a \ln t}{(1+t^2)^b} \, dt.$$

In particular,

$$(5.11) \qquad \frac{d}{da} f(a,b) \Big|_{a=0} = \int_0^\infty \frac{\ln t \, dt}{(1+t^2)^b}.$$

The change of variables $s = t^2$ gives

$$(5.12) \qquad f(a,b) = \frac{1}{2} \int_0^\infty \frac{s^{(a-1)/2} \, ds}{(1+s)^b}.$$

Now use the integral representation

$$(5.13) \qquad B(u,v) = \int_0^\infty \frac{s^{u-1} \, ds}{(1+s)^{u+v}}$$

(given in **8.380.3** in [**3**] and proved in [**5**]) with $u = (a+1)/2$ and $v = b - (a+1)/2$ to obtain

$$(5.14) \qquad f(a,b) = B\left(\frac{a+1}{2}, b - \frac{a+1}{2}\right) = \frac{\Gamma((a+1)/2)\,\Gamma(b - (a+1)/2)}{\Gamma(b)}.$$

Therefore

$$(5.15) \qquad \begin{aligned} \int_0^\infty \frac{\ln t \, dt}{(1+t^2)^b} &= \frac{d}{da} f(a,b) \Big|_{a=0} \\ &= \frac{1}{2\Gamma(b)} \left(\Gamma'((a+1)/2) - \Gamma'(b - (a+1)/2)\right) \Big|_{a=0}. \end{aligned}$$

Now use the relation $\Gamma'(x) = \psi(x)\Gamma(x)$ to obtain the result. $\qquad \square$

The case $b = n \in \mathbb{N}$ requieres the value

$$(5.16) \qquad \Gamma(n + \tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!},$$

and

$$(5.17) \qquad \psi(n + \tfrac{1}{2}) = -\gamma + 2\ln 2 - 2 \sum_{k=1}^n \frac{1}{2k-1} = -\gamma - 2\ln 2 + 2H_{2n} - H_n,$$

that appears in **8.366.3**. Here H_n is the n-th harmonic number. The reader will find a proof of this evaluation in [**2**], page 212. A proof of (5.16) appears as Exercise 10.1.3 on page 190 of [**2**].

Corollary 5.2. Let $n \in \mathbb{N}$. Then

$$(5.18) \qquad \int_0^\infty \frac{\ln x \, dx}{(1 + x^2)^{n+1}} = -\frac{\pi}{2^{2n+1}} \binom{2n}{n} \sum_{k=1}^n \frac{1}{2k - 1}.$$

We now provide a proof of Entry **4.231.7** in [**3**]:

$$\int_0^\infty \frac{\ln x \, dx}{(a^2 + b^2 x^2)^n} = \frac{\Gamma(n - \frac{1}{2})\sqrt{\pi}}{4(n-1)! \, a^{2n-1} b} \left(2\ln\left(\frac{a}{2b}\right) - \gamma - \psi(n - \tfrac{1}{2}) \right).$$

Taking the factor b^2 out of the parenthesis on the left and letting $c = a/b$ yields the equivalent form

$$\int_0^\infty \frac{\ln x \, dx}{(c^2 + x^2)^n} = \frac{\Gamma(n - \frac{1}{2})\sqrt{\pi}}{4(n-1)!} \left(2\ln\left(\frac{c}{2}\right) - \gamma - \psi(n - \tfrac{1}{2}) \right).$$

It is more convenient to replace n by $n + 1$ to obtain

$$\int_0^\infty \frac{\ln x \, dx}{(c^2 + x^2)^{n+1}} = \frac{\Gamma(n + \frac{1}{2})\sqrt{\pi}}{4n!} \left(2\ln\left(\frac{c}{2}\right) - \gamma - \psi(n + \tfrac{1}{2}) \right).$$

Using (5.16) and (5.17) the requested evaluation amounts to

$$(5.19) \qquad \int_0^\infty \frac{\ln x \, dx}{(c^2 + x^2)^{n+1}} = \frac{\pi}{(2c)^{2n+1}} \binom{2n}{n} \left(\ln c - \sum_{k=1}^n \frac{1}{2k - 1} \right).$$

This can be written as

$$(5.20) \qquad \int_0^\infty \frac{\ln x \, dx}{(c^2 + x^2)^{n+1}} = \frac{\pi}{(2c)^{2n+1}} \binom{2n}{n} \left(\ln c - H_n + 2H_{2n} \right).$$

To establish this, make the change of variables $x = ct$ to obtain

$$\int_0^\infty \frac{\ln x \, dx}{(x^2 + c^2)^{n+1}} = \frac{\ln c}{c^{2n+1}} \int_0^\infty \frac{dx}{(t^2 + 1)^{n+1}} + \frac{1}{c^{2n+1}} \int_0^\infty \frac{\ln t \, dt}{(t^2 + 1)^{n+1}}.$$

Using Wallis' formula

$$(5.21) \qquad \int_0^\infty \frac{dt}{(1 + t^2)^{n+1}} = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

the required evaluation now follows from Corollary 5.2.

The special case $n = 0$ yields **4.231.8**:

$$(5.22) \qquad \int_0^\infty \frac{\ln x \, dx}{a^2 + b^2 x^2} = \frac{\pi}{2ab} \ln\left(\frac{a}{b}\right).$$

This evaluation also appears as **4.231.9** in the form

$$(5.23) \qquad \int_0^\infty \frac{\ln px \, dx}{q^2 + x^2} = \frac{\pi}{2q} \ln pq.$$

6. The case of a single purely imaginary pole

In this section we evaluate the integral

(6.1) $$h_{n,2}(a,b) := \int_0^b \frac{\ln t\, dt}{(t^2 + a^2)^{n+1}},$$

for a, $b > 0$ and $n \in \mathbb{N}$. This is the generalization of (4.1) to the case in which the integrand has a multiple pole at $t = ia$. The change of variables $t = ax$ yields

(6.2) $$h_{n,2}(a,b) = a^{-2n-1}g_n(b/a),$$

where

(6.3) $$g_n(x) := \int_0^x \frac{\ln t\, dt}{(t^2 + 1)^{n+1}}.$$

We produce first a recurrence for the companion integral

(6.4) $$f_n(x) := \int_0^x \frac{dt}{(t^2 + 1)^{n+1}}.$$

Lemma 6.1. Let $n \in \mathbb{N}$ and $x > 0$. Then

(6.5) $$2nf_n(x) = (2n-1)f_{n-1}(x) + \frac{x}{(x^2 + 1)^n},$$

and $f_0(x) = \tan^{-1} x$.

PROOF. Integrate by parts. $\qquad\square$

The recurrence (6.5) is now solved using the following result established in [**1**].

Lemma 6.2. Let $n \in \mathbb{N}$ and define $\lambda_j = 2^{2j}\binom{2j}{j}^{-1}$. Suppose the sequence z_n satisfy the recurrence $2nz_n - (2n-1)z_{n-1} = r_n$, with r_n given. Then

(6.6) $$z_n = \frac{1}{\lambda_n}\left(z_0 + \sum_{k=1}^n \frac{\lambda_k r_k}{2k}\right).$$

We conclude with an explicit expression for $f_n(x)$.

Proposition 6.3. Let $n \in \mathbb{N}$. Then

(6.7) $$\int_0^x \frac{dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}}\left(\tan^{-1} x + \sum_{j=1}^n \frac{2^{2j}}{2j\binom{2j}{j}}\frac{x}{(x^2+1)^j}\right).$$

Note 6.4. This expression for f_n appears as **2.148.4** of [**3**] in the alternative form

(6.8) $$\int_0^x \frac{dt}{(1+t^2)^{n+1}} = \frac{x}{2n+1}\sum_{k=1}^n \frac{(2n+1)!!}{(2n-2k+1)!!}\frac{(n-k)!}{2^k n!}\frac{1}{(1+x^2)^{n+1-k}}$$
$$+ \frac{(2n-1)!!}{2^n n!}\tan^{-1} x.$$

We now produce a recurrence for the integral $g_n(x)$.

Lemma 6.5. Let $n \in \mathbb{N}$. Then the integrals $g_n(x)$ satisfy

$$(6.9) \qquad 2ng_n(x) - (2n-1)g_{n-1}(x) = 2n \ln x f_n(x) - [(2n-1)\ln x + 1] f_{n-1}(x).$$

PROOF. Integration by parts yields

$$(6.10) \qquad g_n(x) = f_n(x)\ln x - \int_0^x f_n(t)\frac{dt}{t}.$$

From the recurrence (6.5) we obtain

$$(6.11) \qquad 2n \int_0^x f_n(t)\frac{dt}{t} - (2n-1)\int_0^x f_{n-1}(t)\frac{dt}{t} = f_{n-1}(x).$$

Now replace the integral term from (6.10) to obtain the result. $\qquad\square$

In order to produce a closed-form expression for $g_n(x)$ using Lemma 6.2, we need the initial condition

$$(6.12) \qquad g_0(x) = \int_0^x \frac{\ln t \, dt}{1+t^2}.$$

Lemma 6.6. The function $g_n(x)$ is given by

$$(6.13) \qquad g_n(x) = \sum_{k=0}^{\infty}(-1)^k \binom{n+k}{k}\frac{x^{2k+1}}{2k+1}\left(\ln x - \frac{1}{2k+1}\right).$$

PROOF. The binomial theorem gives

$$(6.14) \qquad (1+t^2)^{-n-1} = \sum_{k=0}^{\infty}(-1)^k \binom{n+k}{k}t^{2k}.$$

The expression for $g_n(x)$ now follows by integrating term by term and the evaluation

$$(6.15) \qquad \int_0^x t^{2k}\ln t \, dt = \left(\ln x - \frac{1}{2k+1}\right)\frac{x^{2k+1}}{2k+1}.$$

$\qquad\square$

In particular, the initial condition $g_0(x)$ of the recurrence (6.9) is given by

$$(6.16) \qquad g_0(x) = \ln x \tan^{-1} x - \mathrm{Ti}_2(x),$$

where

$$(6.17) \qquad \mathrm{Ti}_2(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)^2} = \int_0^x \frac{\tan^{-1} t}{t}\,dt$$

is the tangent integral.

The recurrence (6.9) is now solved using (6.2) to produce

$$
\begin{aligned}
g_n(x) \;=\;& 2^{-2n}\binom{2n}{n}g_0(x) \\[2mm]
&+\; \frac{\binom{2n}{n}}{2^{2n}}\sum_{j=1}^{n}\frac{2^{2j}}{2j\binom{2j}{j}}\left[\{2jf_j(x) - (2j-1)f_{j-1}(x)\}\ln x - f_{j-1}(x)\right].
\end{aligned}
$$

Using the recurrence (6.5), this can be written as

$$(6.18) \qquad g_n(x) = \frac{\binom{2n}{n}}{2^{2n}} \left(g_0(x) + \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \left[\frac{x \ln x}{(x^2+1)^j} - f_{j-1}(x) \right] \right).$$

Solving the recurrence yields:

Theorem 6.7. Let $n \in \mathbb{N}$. Define the rational function

$$(6.19) \qquad p_j(x) = \sum_{k=1}^{j} \frac{2^{2k}}{2k\binom{2k}{k}} \frac{x}{(1+x^2)^k}$$

Then the integral $g_n(x)$ is given by

$$\int_0^x \frac{\ln t \, dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left[g_0(x) + p_n(x) \ln x - \sum_{k=1}^{n} \frac{\tan^{-1} x + p_{k-1}(x)}{2k-1} \right].$$

Note 6.8. The special case $x = 1$ in (6.7) produces

$$(6.20) \qquad \int_0^1 \frac{\ln t \, dt}{(t^2+1)^{n+1}} = 2^{-2n} \binom{2n}{n} \left(G - \sum_{k=1}^{n} \frac{\frac{\pi}{4} + p_{k-1}(1)}{2k-1} \right).$$

The values

$$(6.21) \qquad p_n(1) = \frac{1}{2} \sum_{j=1}^{n} \frac{2^j}{j\binom{2j}{j}}$$

do not admit a closed-form, but they do satisfy the three term recurrence

$$(6.22) \qquad (2n+1)p_{n+1}(1) - (3n+1)p_n(1) + np_{n-1}(1) = 0.$$

The reader is invited to verify the expansion

$$(6.23) \qquad \sum_{k=1}^{\infty} \frac{x^k}{k\binom{2k}{k}} = \frac{2\sqrt{x} \sin^{-1}(\sqrt{x}/2)}{\sqrt{4-x}},$$

from which it follows that

$$(6.24) \qquad \sum_{k=1}^{\infty} \frac{2^k}{k\binom{2k}{k}} = \frac{\pi}{2}.$$

An alternative derivation. Integration by parts produces

$$(6.25) \qquad \int_0^a \frac{\ln s \, ds}{s^2 + b} = \frac{1}{\sqrt{b}} \ln a \tan^{-1} \frac{a}{\sqrt{b}} - \frac{1}{\sqrt{b}} \int_0^{a/\sqrt{b}} \frac{\tan^{-1} x}{x} \, dx.$$

Differentiating n-times with respect to b and using

$$\left(\frac{d}{db} \right)^j \frac{1}{\sqrt{b}} = \frac{(-1)^j (2j)!}{j! 2^{2j} b^{j+1/2}}, \quad \left(\frac{d}{db} \right)^j \frac{1}{a^2 + b} = \frac{(-1)^j j!}{(a^2 + b)^{j+1}},$$

and

$$\left(\frac{d}{db}\right)^j \tan^{-1}\frac{a}{\sqrt{b}} = (-1)^j(j-1)!a\sum_{k=0}^{j-1}\frac{\binom{2k}{k}}{2^{2k+1}b^{k+1/2}(a^2+b)^{j-k}},$$

we obtain

$$\int_0^a \frac{\ln s\, ds}{(s^2+b)^{n+1}} = \ln a \tan^{-1}\left(\frac{a}{\sqrt{b}}\right)F_n(b) + \frac{a\ln a}{2}\sum_{k=1}^n \frac{F_{n-k}(b)}{k}\sum_{j=0}^{k-1}\frac{F_j(b)}{(a^2+b)^{k-j}}$$

$$- F_n(b)\int_0^a \frac{\tan^{-1}x}{x}dx - \frac{1}{2}\sum_{k=1}^n \frac{F_{n-k}(b)}{k}\sum_{j=0}^{j-1}F_j(b)\int_0^a \frac{dt}{(t^2+b)^{k-j}}$$

with $F_j(b) = 2^{-2j}b^{-j-1/2}\binom{2j}{j}$. The last integral in this expression can be evaluated using Proposition 6.3 to produce a generalization of Theorem 6.7. We have replaced the parameter b by b^2 to produce a cleaner formula.

Theorem 6.9. Let $a, b \in \mathbb{R}$ with $a > 0$ and $n \in \mathbb{N}$. Introduce the notation

(6.26)
$$F_j(b) = \frac{\binom{2j}{j}}{2^{2j}b^{2j+1}}.$$

Then

$$\int_0^a \frac{\ln s\, ds}{(s^2+b^2)^{n+1}} = F_n(b)\ln a\tan^{-1}(a/b) + \frac{a\ln a}{2}\sum_{k=1}^n \frac{F_{n-k}(b)}{k}\sum_{j=0}^{k-1}\frac{F_j(b)}{(a^2+b^2)^{k-1}}$$

$$- F_n(b)\int_0^{a/b}\frac{\tan^{-1}x}{x}dx - \frac{1}{2}\tan^{-1}(a/b)\left(\sum_{k=1}^n \frac{F_{n-k}(b)}{k}\sum_{j=0}^{k-1}F_j(b)F_{k-j-1}(b)\right)$$

$$- \frac{a}{4\sqrt{b}}\sum_{k=1}^n \frac{F_{n-k}(b)}{k}\sum_{j=0}^{k-1}F_j(b)F_{k-j-1}(b)\sum_{r=1}^{k-j-1}\frac{1}{rF_r(b)(a^2+b^2)^r}.$$

7. Some trigonometric versions

In this section we provide trigonometric versions of some of the evaluations provided in the previous sections. Many of these integrals correspond to special values of the *Lobachevsky function* defined by

(7.1)
$$L(x) := -\int_0^x \ln\cos t\, dt.$$

This appears as entry **8.260** in [**3**] and also as **4.224.4**. The change of variables $t = \pi/2 - x$ provides a proof of **4.224.1**:

(7.2)
$$\int_0^x \ln\sin t\, dt = L\left(\pi/2 - x\right) - L\left(\pi/2\right).$$

The special value

(7.3)
$$L\left(\pi/2\right) = \int_0^{\pi/2}\ln\sin x\, dx = -\frac{\pi}{2}\ln 2,$$

appears as **4.224.3**. The change of variables $t = \frac{\pi}{2} - x$ yields **4.224.6**:

$$(7.4) \qquad \int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2.$$

To establish these evaluations, observe that, by symmetry,

$$
\begin{aligned}
2 \int_0^{\pi/2} \ln \sin x \, dx &= \int_0^{\pi/2} \ln \sin x \, dx + \int_0^{\pi/2} \ln \cos x \, dx \\
&= \int_0^{\pi/2} \ln \left(\frac{\sin 2x}{2} \right) dx \\
&= \int_0^{\pi/2} \ln(\sin 2x) \, dx - \frac{\pi}{2} \ln 2.
\end{aligned}
$$

Now let $t = 2x$ in the last integral to obtain the result.

Combining (7.1) and (7.2) we obtain **4.227.1**

$$(7.5) \qquad \int_0^u \ln \tan x \, dx = L(u) + L(\pi/2 - u) + \frac{\pi}{2} \ln 2.$$

The identity (1.6) and the evaluation (5.2) yield the value of **4.227.2**:

$$(7.6) \qquad \int_0^{\pi/4} \ln \tan t \, dt = -G.$$

Now observe that

$$(7.7) \qquad \int_0^{\pi/4} \ln \tan t \, dt = \int_0^{\pi/4} \ln \sin t \, dt - \int_0^{\pi/4} \ln \cos t \, dt = -G$$

and

$$(7.8) \qquad \int_0^{\pi/2} \ln \sin t \, dt = \int_0^{\pi/4} \ln \sin t \, dt + \int_0^{\pi/4} \ln \cos t \, dt = -\frac{\pi}{2} \ln 2.$$

Solving this system of equations yields

$$(7.9) \qquad \int_0^{\pi/4} \ln \sin t \, dt = -\frac{\pi}{4} \ln 2 - \frac{G}{2}$$

that appears as **4.224.2** in [**3**] and

$$(7.10) \qquad \int_0^{\pi/4} \ln \cos t \, dt = -\frac{\pi}{4} \ln 2 + \frac{G}{2}$$

that appears as **4.224.5**.

We now make use of the identity

$$(7.11) \qquad \cos x - \sin x = \sqrt{2} \cos(x + \pi/4)$$

to obtain

$$\int_0^{\pi/4} \ln(\cos x - \sin x)\, dx \;=\; \frac{\pi}{8} + \int_0^{\pi/2} \ln \cos(x + \pi/4)$$

$$=\; \frac{\pi}{8} \ln 2 + \int_0^{\pi/2} \ln \cos t\, dt - \int_0^{\pi/4} \ln \cos t\, dt.$$

The first integral is given in (7.4) as $-\frac{\pi}{2} \ln 2$ and the second one as $-\frac{\pi}{4} \ln 2 + \frac{G}{2}$ in (7.10). We conclude with the evaluation of **4.225.1**

(7.12) $$\int_0^{\pi/4} \ln(\cos x - \sin x)\, dx = -\frac{\pi}{8} \ln 2 - \frac{G}{2}.$$

A similar analysis produces **4.225.2**:

(7.13) $$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8} \ln 2 + \frac{G}{2}.$$

These evaluations can be used to produce **4.227.9**:

(7.14) $$\int_0^{\pi/4} \ln(1 + \tan x)\, dx = \frac{\pi}{8} \ln 2,$$

and **4.227.11**:

(7.15) $$\int_0^{\pi/4} \ln(1 - \tan x)\, dx = \frac{\pi}{8} \ln 2 - G.$$

To prove these formulas, simply write

(7.16) $$\ln(1 \pm \tan x) = \ln(\cos \pm \sin x) - \ln \cos x.$$

The same type of calculations provide verification of **4.227.13**

(7.17) $$\int_0^{\pi/4} \ln(1 + \cot x)\, dx = \frac{\pi}{8} \ln 2 + G$$

and **4.227.14**

(7.18) $$\int_0^{\pi/4} \ln(\cot x - 1)\, dx = \frac{\pi}{8} \ln 2.$$

The next example of this type is **4.227.15**:

(7.19) $$\int_0^{\pi/4} \ln(\tan x + \cot x)\, dx = \frac{\pi}{2} \ln 2,$$

This is evaluated by writing the integral as

(7.20) $$-\int_0^{\pi/4} \ln(\sin x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx = \frac{\pi}{2} \ln 2,$$

using (7.9) and (7.10).

The evaluation of **4.227.10**

(7.21) $$\int_0^{\pi/2} \ln(1 + \tan x)\, dx = \frac{\pi}{4} \ln 2 + G,$$

follows from the integrals evaluated here. Indeed,

$$\begin{aligned}
\int_0^{\pi/2} \ln(1 + \tan x)\, dx &= \int_0^{\pi/2} \ln(\sin x + \cos x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx \\
&= 2\int_0^{\pi/4} \ln(\sin x + \cos x)\, dx - \int_0^{\pi/2} \ln(\cos x)\, dx \\
&= 2\left(-\frac{\pi}{8}\ln 2 + \frac{G}{2}\right) + \frac{\pi}{2}\ln 2,
\end{aligned}$$

where we have used (7.4) and (7.13).

The identity (5.5) yields

$$(7.22) \qquad\qquad \int_0^{\pi/2} \ln \tan t\, dt = 0.$$

The apparent generalization

$$(7.23) \qquad\qquad \int_0^{\pi/2} \ln(a \tan t)\, dt = \frac{\pi}{2}\ln a,$$

with $a > 0$, appears as **4.227.3**.

The evaluations (7.9) and (7.10) can be brought back into rational form. The change of variables $t = \tan^{-1} u$ produces from (7.10):

$$(7.24) \qquad\qquad -\frac{\pi}{4}\ln 2 + \frac{G}{2} = -\frac{1}{2}\int_0^1 \frac{\ln(1 + u^2)}{1 + u^2}\, du.$$

We have obtained a proof of **4.295.5**:

$$(7.25) \qquad\qquad \int_0^1 \frac{\ln(1 + x^2)}{1 + x^2}\, dx = \frac{\pi}{2}\ln 2 - G.$$

The change of variables $t = 1/x$ and (5.4) yield **4.295.6**:

$$(7.26) \qquad\qquad \int_1^\infty \frac{\ln(1 + t^2)}{1 + t^2}\, dt = \frac{\pi}{2}\ln 2 + G.$$

There are many other integrals that may be evaluated by the methods reported here. For instance, integration by parts yields

$$(7.27) \qquad\qquad \int_0^x t \cot t\, dt = x \ln \sin x - \int_0^x \ln \sin t\, dt.$$

Using (7.2) we obtain

$$(7.28) \qquad\qquad \int_0^x t \cot t\, dt = x \ln \sin x - L\left(\frac{\pi}{2} - x\right) + \frac{\pi}{2}\ln 2.$$

In particular, from $L(0) = 0$, we obtain **3.747.7**:

$$(7.29) \qquad\qquad \int_0^{\pi/2} \frac{t\, dt}{\tan t} = \frac{\pi}{2}\ln 2.$$

The change of variables $u = \sin x$ produces from here the evaluation of **4.521.1**:

$$(7.30) \qquad \int_0^1 \frac{\text{Arcsin } u}{u} \, du = \frac{\pi}{2} \ln 2.$$

Further evaluations will be reported in a future publication.

References

[1] T. Amdeberhan, L. Medina, and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 5: Some trigonometric integrals. *Scientia*, 15:47–60, 2007.

[2] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.

[3] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[4] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 2: Elementary logarithmic integrals. *Scientia*, 14:7–15, 2007.

[5] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 6: The beta function. *Scientia*, 16:9–24, 2008.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `tamdeber@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `vhm@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `jrosenbe@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `astraub@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
E-mail address: `pwhitwor@tulane.edu`