AN INTEGRAL HIDDEN IN GRADSHTEYN AND RYZHIK

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Abstract. We provide a closed-form expression for the integral

$$N_{0,4}(a;m) \quad := \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

where $m \in \mathbb{N}$ and $a \in (-1,\infty)$:

$$\begin{aligned} N_{0,4}(a;m) &= \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m^{(m+1/2,-m-1/2)}(a) \\ &= \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \times \sum_{k=0}^m 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \end{aligned}$$

Here $P_m^{(m+1/2,-m-1/2)}(a)$ is the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with parameters $\alpha = m+\frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$.

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1. Introduction

We prove

$$N_{0,4}(a;m) := \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.1}$$

where $m \in \mathbb{N}$, $a \in (-1,\infty)$ and

$$P_m(a) := 2^{-2m} \sum_{k=0}^m 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{1.2}$$

The polynomial $P_m(a)$ is an example of the Jacobi family

$$P_m^{(\alpha,\beta)}(a) := \sum_{k=0}^m (-1)^{m-k}\binom{m+\beta}{m-k}\binom{m+k+\alpha+\beta}{k}\left(\frac{a+1}{2}\right)^k \tag{1.3}$$

with parameters $\alpha = m+\frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$. The parameters α and β, usually constants, dependent on m. For small values of m, the integral $N_{0,4}(a;m)$ can be computed by a symbolic language or by a reduction formula ((1.5) below) found in [2]. We were surprised not to find an explicit

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evaluation for $N_{0,4}(a;m)$ in [2] and by the amount of time required to compute it symbolically (e.g. by Mathematica 3.0) for relatively small m (see Section 4). The proof of (1.2) is based on a reduction of $N_{0,4}(a;m)$ to its hypergeometric form. This result is *implicit* in [2] 3.252.11, thus motivating the title of this paper.

Integrals of the type

$$\int_0^\infty \frac{x^\mu \, dx}{(ax^2+2bx+c)^\nu}$$

are discussed in section 3.252 of [2]. In most cases $\mu, \nu \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. A notable exception is 3.252.11 mentioned above, which is equivalent to

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{(x^2+2ax+1)^{\mu+1/2}} = \frac{2^\mu \Gamma(1+\mu) B(-\nu+2\mu+1,\nu) P_{\mu-\nu}^{-\mu}(a)}{(a^2-1)^{\mu/2}}. \tag{1.4}$$

Expressing the associated Legendre function $P_\nu^\mu(x)$ in its hypergeometric form

$$P_\nu^\mu(a) = \frac{1}{\Gamma(1-\mu)} \left(\frac{a+1}{a-1} \right)^{\mu/2} {}_2F_1\left[-\nu,\nu+1;1-\mu;\tfrac{1-a}{2}\right]$$

we obtain

$$\begin{aligned} \int_0^\infty \frac{x^{\nu-1}\,dx}{(x^2+2ax+1)^{\mu+1/2}} &= \left(\frac{2}{a+1}\right)^\mu B\left(2\mu+1-\nu,\nu\right) \times \\ & \qquad {}_2F_1\left[\nu-\mu,1+\mu-\nu;1+\mu;\tfrac{1-a}{2}\right], \end{aligned}$$

where $a>-1$, $\nu>0$, and $\nu-2\mu<1$.

The only explicit appearance of $N_{0,4}(a;m)$ in [2] is the recursion 2.161.5, which yields

(1.5)
$$\begin{aligned} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} &= \frac{(4m-3)a}{4m(a^2-1)} \int_0^\infty \frac{x^2dx}{(x^4+2ax^2+1)^m} \\ &+ \frac{4m(a^2-1)+1-2a^2}{4m(a^2-1)} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^m}. \end{aligned}$$

This is useful for small values of m but inefficient for large m. We present a proof of (1.5) in Section 3. This recursion for the integrals $N_{0,4}(a;m)$ produces a recursion for the polynomials $P_m(a)$ ((3.3) below). These polynomials *do not* satisfy the standard recursion relation ((3.4) below) for the classical Jacobi polynomials.

The explicit evaluation of $N_{0,4}(a;m)$ is the basic instrument in an algorithm for the integration of even rational functions described in [1].

2. The proof

Theorem 2.1. *Let* $a \in (-1, \infty)$ *and* $m \in \mathbb{N}$. *Then*

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \times \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k.$$

Proof. The change of variables $x \to \sqrt{x}$ and (1.4) produce

$$\begin{aligned} N_{0,4}(a;m) &= 2^{m-1/2}(a^2-1)^{-m/2-1/4}\Gamma\left(m+\tfrac{3}{2}\right) B\left(2m+\tfrac{3}{2},\tfrac{1}{2}\right) P_m^{-m-1/2}(a) \\ &= 2^{m-1/2}\,(a+1)^{-(m+\frac{1}{2})}\,B(2m+\tfrac{3}{2},\tfrac{1}{2})\,{}_2F_1\left[-m,m+1;m+\tfrac{3}{2};\tfrac{1-a}{2}\right]. \end{aligned}$$

Now recognize the ${}_2F_1$ as a Jacobi polynomial

$${}_2F_1\left[-m,m+1;m+\tfrac{3}{2};\tfrac{1-a}{2}\right] = \frac{m!\,\Gamma\left(m+\frac{3}{2}\right)}{\Gamma\left(2m+\frac{3}{2}\right)} P_m^{(m+1/2,-m-1/2)}(a),$$

so that

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m^{(m+1/2,-m-1/2)}(a). \tag{2.1}$$

Using (1.3) we obtain

$$P_m^{(m+1/2,-m-1/2)}(a) = 2^{-2m}\sum_{k=0}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k.$$

□

Introduce the notation

$$N_{n,4}(a,b,c;m) := \int_0^\infty \frac{x^{2n}}{(bx^4+2ax^2+c)^{m+1}}\,dx$$

and set $N_{n,4}(a;m) := N_{n,4}(a,1,1;m)$.

Theorem 2.2. *Let* $b>0,\ c>0,\ a>-\sqrt{bc},\ m\in\mathbb{N}$ *and* $0 \le n \le 2m+1$. *Define*

$$T_1(a,b,c;m,n) := \left(c\left(\frac{c}{b}\right)^{m-n}\left(8(a+\sqrt{bc})\right)^{2m+1}\right)^{1/2}$$

and

$$T_2(m,n) := |m-n| - \lfloor \tfrac{n}{m+1} \rfloor.$$

Then

$$N_{n,4}(a,b,c;m) = \frac{\pi}{T_1(a,b,c;m,n)} \times \sum_{k=0}^{T_2(m,n)} 2^k\binom{2m-2k}{m-k}\binom{m-n+k}{2k}\binom{2k}{k}\binom{m}{k}^{-1}\left(\frac{a}{\sqrt{bc}}+1\right)^k.$$

Proof. Let $u = (b/c)^{1/4}x$. Then Theorem 2.1 gives an expression for $N_{0,4}(a,b,c;m)$. The result now follows by differentiation. □

Corollary 2.3. *Suppose $0 \leq n \leq m$. Then*

$$N_{n,4}(a;m) = \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \times \times \sum_{k=0}^{m-n} 2^k \binom{2m-2k}{m-k}\binom{m-n+k}{2k}\binom{2k}{k}\binom{m}{k}^{-1}(a+1)^k.$$

The sum for the case $m+1 \leq n \leq 2m+1$ is obtained by the symmetry relation

$$N_{n,4}(a;m) = N_{2m+1-n,4}(a;m) \tag{2.2}$$

which follows from the change of variables $x \to 1/x$.

Example 1.

$$\int_0^\infty \frac{x^6\,dx}{(2x^4+2x^2+3)^{11}} = \frac{11\pi(14229567+4937288\sqrt{6})}{440301256704(1+\sqrt{6})^{21/2}}.$$

3. A recursion

In this section we prove (1.5). The argument is based on Hermite's reduction procedure for the indefinite integration of rational functions.

Let $V(x) = x^4 + 2ax^2 + 1$. Then V and V' have no common factor so the Euclidean algorithm produces polynomials B and C such that

$$-\frac{1}{m} = CV + BV'. \tag{3.1}$$

Indeed, a simple calculation yields

$$B(x) = -\frac{1}{4m}\frac{1}{a^2-1}\left((1-2a^2)x - ax^3\right) \text{ and } C(x) = -\frac{1}{m}\left(1+\frac{a}{a^2-1}x^2\right).$$

Divide (3.1) by V^{m+1} and integrate from 0 to ∞ to produce

$$N_{0,4}(a;m) = \left(1+\frac{1-2a^2}{4m(a^2-1)}\right)N_{0,4}(a;m-1) + \frac{(4m-3)a}{4m(a^2-1)}N_{1,4}(a;m-1),$$

which is (1.5). This recursion can be also be written as

$$\begin{aligned} N_{0,4}(a;m) &= \left(1+\frac{1-2a^2}{4m(a^2-1)}\right)N_{0,4}(a;m-1) \\ &- \frac{(4m-3)a}{8m(m-1)(a^2-1)}\frac{d}{da}N_{0,4}(a;m-2). \end{aligned} \tag{3.2}$$

Proposition 3.1. *The polynomials $P_m(a)$ satisfy*

$$
\begin{aligned}
P_m(a) &= \frac{(2m-3)(4m-3)a}{4m(m-1)(a-1)} P_{m-2}(a) - \frac{(4m-3)a(a+1)}{2m(m-1)(a-1)} \frac{d}{da} P_{m-2}(a) \\
&+ \frac{4m(a^2-1)+1-2a^2}{2m(a-1)} P_{m-1}(a). \qquad (3.3)
\end{aligned}
$$

Proof. Use (1.1) in (3.2). □

The special values $P_m(1) = 2^{-2m}\binom{4m+1}{2m}$ and $P'_m(1) = \frac{m(m+1)}{2m+3} P_m(1)$ computed directly from the integral $N_{0,4}(a;m)$ show that the right-hand side of (3.3) is, in spite of its appearance, a polynomial in a.

The Jacobi polynomials $P_m^{(\alpha,\beta)}(a)$ in (1.3) satisfy the recursion

$$
\begin{aligned}
& 2(m+1)(\gamma+m+1)(\gamma+2m) P_{m+1}^{(\alpha,\beta)}(a) \qquad (3.4) \\
&= (\gamma+2m+1)\left(\alpha^2-\beta^2+a(\gamma+2m+2)(\gamma+2m)\right) P_m^{(\alpha,\beta)}(a) \\
&\quad -2(\alpha+m)(\beta+m)(\gamma+2m+2) P_{m-1}^{(\alpha,\beta)}(a),
\end{aligned}
$$

where $\gamma := \alpha+\beta$. This *is not* satisfied by $P_m(a)$ because in the derivation of (3.4) the parameters α and β are assumed to be independent of m.

4. CPU times

We compute the values of

$$
N_{0,4}(4;m) = \int_0^\infty \frac{dx}{(x^4+8x^2+1)^{m+1}} \qquad (4.1)
$$

for several values of m to illustrate the power of (1.1). The calculations were done on a SUN Ultra 1 using Mathematica 3.0. The table compares the CPU times of the direct calculation (Time 1) with the CPU times (Time 2) when formula (1.1) is used. The times of the direct calculation do not improve significantly if the quartic is factored and the integrand expanded in partial fractions.

m	Time 1	Time 2
0	24.6	0.0
25	40.92	0.06
50	224.37	0.06
75	450.17	0.11
100	1369.95	0.16

Table 1. Calculation of $N_{0,4}(4,m)$.

5. An example of degree eight

We use Theorem 2.2 to evaluate the integral

$$N_{0,8}(a_1,a_2;m) \quad := \quad \int_0^\infty \frac{dx}{(x^8+a_2x^6+2a_1x^4+a_2x^2+1)^{m+1}}.$$

The denominator is a *symmetric* polynomial $D_8(x)$ of degree 8, i.e. it satisfies $D_8(1/x)=x^{-8}D_8(x)$. This type of polynomial is at the center of the algorithm developed in [1].

Theorem 5.1. *Let $m\in\mathbb{N}$ and $a_1 >$ $\max\left\{-a_2-1, -\frac{1}{8}(a_2^2+8)\times \operatorname{sign}(a_2+4)\right\}$. Then*

$$N_{0,8}(a_1,a_2;m) \quad = \quad 2^{-m}\sum_{j=0}^{2m+1}\binom{4m+3}{2j}\sum_{k=0}^{2m-j+1}\binom{2m-j+1}{k}N_{k,4}(a_2+4,a_1+a_2+1,8;m).$$

Proof. The substitutions $x=\tan\theta$ and $u=2\theta$ yield

$$N_{0,8}(a_1,a_2;m) \quad = \quad 2\int_0^{\pi/2}\frac{(1+\cos 2\theta)^{4m+3}}{\Phi^{m+1}(\theta)}\,d\theta,$$

where, with $c=\cos 2\theta$,

$$\Phi(\theta) \quad := \quad (1-c)^4+a_2(1-c)^3(1+c)+2a_1(1-c^2)^2+a_2(1-c)(1+c)^3+(1+c)^4.$$

Thus, with

$$\Psi(u) \quad := \quad (a_1+a_2+1)+2(3-a_1)\cos^2 u+(a_1-a_2+1)\cos^4 u,$$

we have

$$\begin{aligned} N_{0,8}(a_1,a_2;m) &= 2^{-m-1}\int_0^\pi \frac{(1+\cos u)^{4m+3}\,du}{\Psi^{m+1}(u)} \\ &= 2^{-m-1}\sum_{j=0}^{2m+1}\binom{4m+3}{2j}\int_0^\pi\frac{\cos^{2j}u}{\Psi^{m+1}(u)}\,du, \qquad (5.1)\end{aligned}$$

where in the last step we have used the fact that

$$\int_0^\pi \frac{\cos^j u\,du}{\Psi^{m+1}(u)} \quad = \quad 0 \text{ for odd } j.$$

The substitutions $v = 2u$ and $x = \tan v/2$ now give

$$
\begin{aligned}
I_{m,\nu} &:= 2^{-m-1}\int_0^{\pi} \frac{\cos^{2j} u\, du}{\Psi^{m+1}(u)} \qquad (5.2)\\
&= 2^{m+1-j}\int_0^{\pi} \frac{(1+\cos 2u)^j\ du}{[4(a_1+a_2+1)+4(3-a_1)(1+\cos 2u)+(a_1-a_2+1)(1+\cos 2u)^2]^{m+1}}\\
&= 2^{m-j}\int_0^{2\pi} \frac{(1+\cos v)^j\ dv}{[(a_1+3a_2+17)-2(a_1+a_2-7)\cos v+(a_1-a_2+1)\cos^2 v]^{m+1}}\\
&= 2^{m-j+1}\int_0^{\pi} \frac{(1+\cos v)^j\ dv}{[(a_1+3a_2+17)-2(a_1+a_2-7)\cos v+(a_1-a_2+1)\cos^2 v]^{m+1}}\\
&= 2^{-m}\int_0^{\infty} \frac{(x^2+1)^{2m-j+1}\ dx}{U^{m+1}(x)},
\end{aligned}
$$

where $U(x) = (a_1+a_2+1)x^4 + 2(a_2+4)x^2 + 8$. Combining (5.1) and (5.2) yields

$$
\begin{aligned}
N_{0,8}(a_1,a_2;m) &= 2^{-m}\sum_{j=0}^{2m+1}\binom{4m+3}{2j}\int_0^{\infty}\frac{(x^2+1)^{2m-j+1}\ dx}{U^{m+1}(x)}\\
&= 2^{-m}\sum_{j=0}^{2m+1}\binom{4m+3}{2j}\sum_{k=0}^{2m-j+1}\binom{2m-j+1}{k}\times\int_0^{\infty}\frac{x^{2k}\ dx}{U^{m+1}(x)}\\
&= 2^{-m}\sum_{j=0}^{2m+1}\binom{4m+3}{2j}\sum_{k=0}^{2m-j+1}\binom{2m-j+1}{k}N_{k,4}(a_2+4,a_1+a_2+1,8;m),
\end{aligned}
$$

as claimed. $\square$

Example 2.

$$
\int_0^{\infty}\frac{dx}{(x^8+5x^6+14x^4+5x^2+1)^4} = \frac{(14325195794+2815367209\sqrt{26})\pi}{14623232(9+2\sqrt{26})^{7/2}}.
$$

6. Update. January 2009

The paper [1] appeared as [4]. Many proofs of the evaluation of $N_{0,4}(a;m)$ and the explicit formula for the coefficients of $P_m(a)$ have been summarized in [3].

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