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On Some Integrals Involving the Hurwitz Zeta Function: Part 2

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Abstract. We establish a series of indefinite integral formulae involving the Hurwitz zeta function and other elementary and special functions related to it, such as the Bernoulli polynomials, $\ln \sin(\pi q)$, $\ln \Gamma(q)$ and the polygamma functions. Many of the results are most conveniently formulated in terms of a family of functions $A_k(q) := k\zeta'(1-k, q)$, $k \in \mathbb{N}$, and a family of polygamma functions of negative order, whose properties we study in some detail.

Key words: Hurwitz zeta function, polylogarithms, loggamma, integrals

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1. Introduction

The Hurwitz zeta function, defined by

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{1.1}$$

for $z \in \mathbb{C}$, $\operatorname{Re} z > 1$ and $q \neq 0, -1, -2, \ldots$, admits the integral representation

$$\zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{e^{-qt}}{1 - e^{-t}} t^{z-1} dt, \tag{1.2}$$

where $\Gamma(z)$ is Euler's gamma function, which is valid for $\operatorname{Re} z > 1$ and $\operatorname{Re} q > 0$, and can be used to prove that $\zeta(z, q)$ has an analytic extension to the whole complex plane except for a simple pole at $z = 1$.

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For $\operatorname{Re} z < 0$, $\zeta(z, q)$ admits the following Fourier representation, originally derived by Hurwitz, in the range $0 \le q \le 1$:

$$\zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi q n)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi q n)}{n^{1-z}} \right]. \tag{1.3}$$

This representation was used in [5] to obtain several definite integral formulae involving $\zeta(z, q)$. A derivation of (1.3) can be found in [11, p. 268]. An alternative proof, based upon the representation

$$\zeta(z, q) = \frac{q^{1-z}}{z-1} + \frac{q^{-z}}{2} - z \int_0^{\infty} \frac{\{t\} - \frac{1}{2}}{(t+q)^{z+1}}\, dt, \tag{1.4}$$

where $\{t\}$ is the fractional part of t, has been given by Berndt [4]. The expression (1.4) is employed in [4] to give short proofs of several classical formulae, including Lerch's beautiful expression

$$\ln \Gamma(q) = \zeta'(0, q) - \zeta'(0). \tag{1.5}$$

In this paper we continue the work, initiated in [5], on the explicit evaluation of integrals involving $\zeta(z, q)$. Special cases of $\zeta(z, q)$ include the Bernoulli polynomials,

$$B_m(q) = -m\, \zeta(1-m, q), \quad m \in \mathbb{N}, \tag{1.6}$$

defined by their generating function

$$\frac{t e^{qt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(q) \frac{t^m}{m!}, \tag{1.7}$$

and given explicitly in terms of the Bernoulli numbers B_k by

$$B_m(q) = \sum_{k=0}^{m} \binom{m}{k} B_k q^{m-k}; \tag{1.8}$$

the digamma function,

$$\psi(q) := \frac{d}{dq} \ln \Gamma(q) = \lim_{z \to 1} \left[\frac{1}{z-1} - \zeta(z, q) \right]; \tag{1.9}$$

and the polygamma functions,

$$\psi^{(m)}(q) = (-1)^{m+1}\, m!\, \zeta(m+1, q), \quad m \in \mathbb{N}, \tag{1.10}$$

defined by

$$\psi^{(m)}(q) := \frac{d^m}{dq^m} \psi(q), \quad m \in \mathbb{N}. \tag{1.11}$$

An important property of the Hurwitz zeta function, which will be essential for the indefinite integral evaluations presented in Section 2, is the following:

$$\frac{\partial}{\partial q}\,\zeta(z,q) = -z\,\zeta(z+1,q). \tag{1.12}$$

The rest of this paper is organized as follows. In Section 2 we consider the evaluation of indefinite integrals of functions of the form $f(q)\zeta(z,a+bq)$, using a simple integration by parts approach. In Section 3 we introduce and study some of the properties of two families of functions related to the first derivative with respect to the argument z of the Hurwitz zeta function $\zeta(z,q)$, evaluated at z equal to nonpositive integers. These functions appear in connection to the indefinite integrals involving polygamma and negapolygamma functions, as well as $\ln\Gamma(q)$ and $\ln\sin(\pi q)$, considered in Section 4. These, in turn, are derived from the formulae obtained in Section 2 either by direct differentiation or by taking the appropriate limits. Finally, in Section 5 we use some of the indefinite integral formulae to rederive some of the definite integral evaluations obtained in Ref. [5] and to present some new analogous formulae.

2. The evaluation of indefinite integrals

In this section we discuss a method to evaluate primitives of functions of the form $f(q)\zeta(z,a+bq)$. This is illustrated in the cases where f is a polynomial and an exponential function. The resulting evaluations can be taken as a starting point to derive similar formulae involving other special functions in place of $\zeta(z,a+bq)$. For instance, differentiation with respect to the parameter z leads, in view of Lerch's result (1.5), to the evaluation of primitives involving the weight $\ln\Gamma(a+bq)$, and thus also $\ln\sin\pi q$, by virtue of the reflection formula for the gamma function. Also, the limit $z\to m\in\mathbb{N}$ leads to the evaluation of primitives involving the polygamma function $\psi^{(m-1)}(q)$. We shall present these results, from a slightly more general point of view, in Section 4.

Theorem 2.1. *Let $r\in\mathbb{N}$, f be r-times differentiable and $a,\ b\in\mathbb{R}$. Then*

$$\begin{aligned}\int f(q)\zeta(z,a+bq)\,dq &= \sum_{k=1}^{r}(-1)^{k+1}\frac{f^{(k-1)}(q)\,\zeta(z-k,a+bq)}{b^k(1-z)_k}\\ &\quad+\frac{(-1)^r}{b^r(1-z)_r}\int f^{(r)}(q)\,\zeta(z-r,a+bq)\,dq.\end{aligned} \tag{2.1}$$

Proof: Observe that

$$\frac{\partial}{\partial q}\zeta(z-1,a+bq) = b(1-z)\zeta(z,a+bq), \tag{2.2}$$

so that integration by parts yields

$$\int f(q)\zeta(z, a+bq)\,dq = \frac{f(q)\zeta(z-1, a+bq)}{b(1-z)} - \frac{1}{b(1-z)}\int f'(q)\,\zeta(z-1, a+bq)\,dq.$$

The expression (2.1) follows by repeating this procedure. □

We now produce the evaluation of certain indefinite integrals by choosing appropriate functions f in Theorem 2.1.

Example 2.2. Let $n \in \mathbb{N}_0$ and $a,\ b \in \mathbb{R}$. Then the moments of $\zeta(z, q)$ are given by

$$\int q^n \zeta(z, a+bq)\,dq = n!\sum_{j=0}^{n} \frac{(-1)^j q^{n-j}}{b^{j+1}(1-z)_{j+1}(n-j)!}\zeta(z-j-1, a+bq). \tag{2.3}$$

Proof: The case $n = 0$ is simply the known result

$$\int \zeta(z, a+bq)\,dq = \frac{\zeta(z-1, a+bq)}{b(1-z)}. \tag{2.4}$$

For $n \geq 1$, the function $f(q) = q^n$ satisfies $f^{(k-1)}(q) = n!\,q^{n-k+1}/(n-k+1)!$, for $k \leq n$. Then (2.1), with $r = n$, yields

$$\int q^n \zeta(z, a+bq)\,dq = \sum_{k=1}^{n} \frac{(-1)^{k+1} n!\,q^{n-k+1}}{(n-k+1)!\,b^k(1-z)_k}\zeta(z-k, a+bq) + \frac{(-1)^n\,n!}{b^n(1-z)_n}\int \zeta(z-n, a+bq)\,dq,$$

so that (2.3) follows from (2.4). □

In a similar fashion we obtain:

Example 2.3. Let $m \in \mathbb{N}_0$ and $a,\ b,\ c,\ d \in \mathbb{R}$. Then

$$\int B_m(c+dq)\zeta(z, a+bq)\,dq = m!\sum_{j=0}^{m} \frac{(-1)^j d^{\,j} B_{m-j}(c+dq)}{b^{j+1}(1-z)_{j+1}(m-j)!}\zeta(z-j-1, a+bq). \tag{2.5}$$

Proof: Same as the proof for Example 3.2 with

$$\frac{d^{\,k-1}}{dq^{\,k-1}} B_m(c+dq) = \frac{m!\,d^{k-1}}{(m-k+1)!} B_{m-k+1}(c+dq).$$ □

Definition. The family of functions $\mathfrak{F} := \{f_j(q) : j \in \mathbb{N}\}$ is said to be *closed under primitives* if for each j the primitive of $f_j(q)$ can be written as a *finite* linear combination of the elements of $\mathfrak{F}$. Naturally, the family $\mathfrak{F}$ is allowed to depend on a finite number of parameters, as in the next example.

Example 2.4. Example 2.2 shows that

$$\mathfrak{F}_{a,b} := \{P_j(q)\zeta(z-m, a+bq) : j,\ m \in \mathbb{N} \text{ and } P_j \text{ is a polynomial in } q \text{ of degree } j\}$$

is closed under primitives. This follows from

$$\int q^n \zeta(z-m, a+bq)\, dq = n! \sum_{j=0}^{n} \frac{(-1)^j q^{n-j} \zeta(z-m-1-j, a+bq)}{b^{j+1}(m+1-z)_{j+1}(n-j)!}, \tag{2.6}$$

which is a variation of (2.3).

Example 2.5. The moments of the Bernoulli polynomials are given by

$$\int q^n B_m(a+bq)\, dq = \frac{n!\, m!}{(n+m+1)!} \sum_{j=0}^{n} \frac{(-1)^j q^{n-j}}{b^{j+1}} \binom{m+n+1}{n-j} B_{m+j+1}(a+bq). \tag{2.7}$$

Proof: Use the identity (1.6) in (2.3). □

Example 2.6. Let $n \in \mathbb{N}$ be odd. Then

$$\int \zeta(z-n, q)\zeta(z, q)\, dq = \frac{1}{2} \sum_{k=1}^{n} \frac{(z-n)_{k-1}}{(1-z)_k} \zeta(z-k, q)\, \zeta(z-n+k-1, q). \tag{2.8}$$

Proof: The Hurwitz zeta function satisfies

$$\frac{\partial^{k-1}}{\partial q^{k-1}} \zeta(z-n, q) = (-1)^{k-1}(z-n)_{k-1}\zeta(z-n+k-1, q), \tag{2.9}$$

so the result follows from (2.1) with $r = n$, since in that case the integral on the righthand side equals the one on the left-hand side, except for the prefactor $(z-n)_n/(1-z)_n = (-1)^n$. □

Note. In view of the identity

$$\frac{(z-n)_{k-1}}{(1-z)_k} = (-1)^{n+1} \frac{(z-n)_{n-k}}{(1-z)_{n-k+1}},$$

it is easily seen that the terms in the sum on the right-hand side of (2.9) are equal in pairs, except for the central term $k = r$, where $r \in \mathbb{N}$ is defined by $n = 2r - 1$. Therefore we have the alternative formula

$$\int \zeta(z-2r+1,q)\zeta(z,q)\,dq = \frac{(z-2r+1)_{r-1}}{2\,(1-z)_r}\zeta^2(z-r,q) + \sum_{k=1}^{r-1} \frac{(z-2r+1)_{k-1}}{(1-z)_k}\zeta(z-k,q)\zeta(z-(2r-k),q). \tag{2.10}$$

Note. We have been unable to evaluate the integral in Example 2.6 for the case n even. Thus the question of whether the family

$$\mathfrak{F}_z := \{\zeta(z-n,q)\zeta(z-m,q):\ n,m\in\mathbb{N}\} \tag{2.11}$$

is closed under primitives remains to be decided.

Example 2.7. Let $a,\ b\in\mathbb{R}$. Then

$$\int e^q\zeta(z,a+bq)\,dq = e^q\sum_{j=0}^{\infty}\frac{(-1)^j}{b^{j+1}(1-z)_{j+1}}\,\zeta(z-1-j,a+bq). \tag{2.12}$$

Proof: Divide (2.3) by $n!$ and then sum over n to produce

$$\int e^q\zeta(z,a+bq)\,dq = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^j q^{n-j}}{b^{j+1}(1-z)_{j+1}(n-j)!}\,\zeta(z-1-j,a+bq).$$

The result follows by interchanging the order of summation. □

Note. We have been unable to produce a *finite expression* for the integral in (2.12).

Example 2.8. Let $m\in\mathbb{N}$. Then

$$\int e^q B_m(a+bq)\,dq = m!\,e^q(-1)^m\sum_{j=0}^{m}\frac{(-1)^j}{j!}b^{m-j}B_j(a+bq). \tag{2.13}$$

Proof: Use the identity (1.6) and $(m)_{j+1}=(m+j)!/(m-1)!$ in (2.12) to produce

$$\int e^q B_m(a+bq)\,dq = m!\,e^q(-1)^{m+1}\sum_{j=m+1}^{\infty}\frac{(-1)^j}{j!}b^{m-j}B_j(a+bq). \tag{2.14}$$

The generating function (1.7) is now employed to see that the sum from $j=0$ to infinity is independent of q, so it is absorbed into the implicit constant of integration. □

3. The function $A_k(q)$ and negapolygammas

In this section we consider the function

$$A_k(q) := k\,\frac{\partial}{\partial z}\,\zeta(z,q)\bigg|_{z=1-k} \tag{3.1}$$

for $k \in \mathbb{N}$. The function $A_1(q)$ has a simple explicit form,

$$A_1(q) = \zeta'(0,q) = \ln\Gamma(q) + \zeta'(0), \tag{3.2}$$

in view of Lerch's result (1.5). These functions appear in all of the formulae for the indefinite integrals involving the loggamma and the logsine functions studied in Section 4. The derivative of the Hurwitz zeta function has appeared before in connection to integrals of $\ln\Gamma(q)$ [6], and in a number of related contexts, such as the studies of polygamma functions of negative order [1], the Barnes function [2] and the multiple gamma function [10], and other unrelated ones such as the evaluation of sums of the type $\sum_{m\geq 2} z^m\zeta(m,\alpha)$ [7]. Here, we will rediscover, from a more general point of view, the intimate relationship existing between the functions $A_k(q)$ and polygamma functions of negative order, first pointed out in Ref. [6].

The nonelementary behavior of $A_k(q)$ can be considered to be contained in the range $0 \leq q \leq 1$, since for $q > 1$ the value of $A_k(q)$ can be obtained by repeated use of the following result:

Lemma 3.1. *The function $A_k(q)$ satisfies*

$$A_k(q+1) = A_k(q) + kq^{k-1}\ln q. \tag{3.3}$$

Proof: Differentiate both sides of the identity

$$\zeta(z,q) = \frac{1}{q^z} + \zeta(z,q+1) \tag{3.4}$$

with respect to z at $z = 1-k$. □

As an immediate consequence of property (3.3) and definition (3.1) we have

Lemma 3.2. *For $k \geq 2$,*

$$A_k(0) = A_k(1) = k\,\zeta'(1-k). \tag{3.5}$$

Proof: Take the limit $q \to 0$ in (3.3) and use the fact that $\zeta(z,1) = \zeta(z)$. □

Lemma 3.3. *For $k \in \mathbb{N}$,*

$$\int_0^1 A_k(q)\,dq = 0. \tag{3.6}$$

Proof: This is a direct consequence of

$$\int_0^1 \zeta(z,q)\,dq = 0, \tag{3.7}$$

valid for $\operatorname{Re} z < 1$. □

Lemma 3.4. *For $k \in \mathbb{N}$, the functions $A_k(q)$ satisfy*

$$A'_{k+1}(q) = (k+1)\left[A_k(q) + \frac{1}{k}B_k(q)\right]. \tag{3.8}$$

Proof: We have

$$\begin{aligned} A'_{k+1}(q) &= (k+1)\frac{\partial}{\partial q}\frac{\partial}{\partial z}\zeta(z,q)\bigg|_{z=-k} = (k+1)\frac{\partial}{\partial z}\frac{\partial}{\partial q}\zeta(z,q)\bigg|_{z=-k} \\ &= (k+1)\frac{\partial}{\partial z}\left[-z\zeta(z+1,q)\right]\bigg|_{z=-k} \\ &= (k+1)\left[A_k(q) + \frac{1}{k}B_k(q)\right]. \end{aligned}$$

□

The following results can sometimes be used to simplify a formula:

$$\zeta'(-2n) = (-1)^n \frac{(2n)!\,\zeta(2n+1)}{2\,(2\pi)^{2n}}, \quad n \in \mathbb{N}, \tag{3.9}$$

$$\zeta'(0) = -\ln\sqrt{2\pi}. \tag{3.10}$$

Lemma 3.5. *For $k \in \mathbb{N}$,*

$$A_k\left(\frac{1}{2}\right) = (-1)^{k-1}B_k 2^{1-k}\ln 2 - (1-2^{1-k})k\,\zeta'(1-k). \tag{3.11}$$

In particular,

$$A_1\left(\frac{1}{2}\right) = -\frac{1}{2}\ln 2, \tag{3.12}$$

$$A_{2n+1}\left(\frac{1}{2}\right) = (-1)^{n+1}\frac{(1-2^{-2n})(2n+1)!\,\zeta(2n+1)}{2(2\pi)^{2n}} \quad n \geq 1. \tag{3.13}$$

Proof: Differentiate $\zeta(z,\frac{1}{2}) = (2^z-1)\,\zeta(z)$ at $z = 1-k$ and use the identity

$$\zeta(1-k) = \frac{(-1)^{k+1}B_k}{k}, \quad k \in \mathbb{N}. \tag{3.14}$$

□

Hurwitz's Fourier representation (1.3) for $\zeta(z, q)$ in the range $0 \le q \le 1$ and negative z is absolutely convergent and thus can be used to directly obtain a Fourier representation for the function $A_k(q)$ in the range $0 \le q \le 1$.

Define

$$C(z, q) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nq)}{n^z} \quad \text{and} \quad S(z, q) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nq)}{n^z}, \tag{3.15}$$

so that

$$\frac{\partial}{\partial z} C(z, q) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^z} \cos(2\pi nq) \quad \text{and} \quad \frac{\partial}{\partial z} S(z, q) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^z} \sin(2\pi nq).$$

Now replace z by $1 - z$ in the Fourier representation (1.3) and differentiate to produce

$$\begin{aligned} \zeta'(1 - z, q) = \frac{2\Gamma(z)}{(2\pi)^z} \Bigg\{ & \left[\Psi(z) \cos\frac{\pi z}{2} + \frac{\pi}{2} \sin\frac{\pi z}{2} \right] C(z, q) \\ & + \left[\Psi(z) \sin\frac{\pi z}{2} - \frac{\pi}{2} \cos\frac{\pi z}{2} \right] S(z, q) \\ & - \cos\frac{\pi z}{2} \frac{\partial}{\partial z} C(z, q) - \sin\frac{\pi z}{2} \frac{\partial}{\partial z} S(z, q) \Bigg\}, \end{aligned} \tag{3.16}$$

where $\Psi(z) := \ln 2\pi - \psi(z)$.

For z a positive integer, the functions $S(z, q)$ and $C(z, q)$ are related to the Bernoulli polynomials (in view of (1.3) and (1.6)) and the Clausen functions. The latter are defined by

$$Cl_{2n}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad n \ge 1 \tag{3.17}$$

and

$$Cl_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}}, \quad n \ge 0. \tag{3.18}$$

One has

$$S(2m + 1, q) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nq)}{n^{2m+1}} = \frac{(-1)^{m+1} (2\pi)^{2m+1}}{2(2m + 1)!} B_{2m+1}(q), \tag{3.19}$$

$$C(2m + 1, q) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nq)}{n^{2m+1}} = Cl_{2m+1}(2\pi q), \tag{3.20}$$

$$S(2m + 2, q) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nq)}{n^{2m+2}} = Cl_{2m+2}(2\pi q), \tag{3.21}$$

$$C(2m + 2, q) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nq)}{n^{2m+2}} = \frac{(-1)^m (2\pi)^{2m+2}}{2(2m + 2)!} B_{2m+2}(q). \tag{3.22}$$

The value $z = 2m + 1$ yields, upon using $\psi(k+1) = -\gamma + H_k$, the expression

$$\begin{aligned} A_{2m+1}(q) &= (H_{2m} - \gamma - \ln 2\pi)\, B_{2m+1}(q) \\ &\quad + (-1)^m \frac{2(2m+1)!}{(2\pi)^{2m+1}} \left[\sum_{n=1}^{\infty} \frac{\ln n}{n^{2m+1}} \sin(2\pi n q) + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n q)}{n^{2m+1}} \right]. \end{aligned} \tag{3.23}$$

Similarly, for $z = 2m + 2$ we find

$$\begin{aligned} A_{2m+2}(q) &= (H_{2m+1} - \gamma - \ln 2\pi)\, B_{2m+2}(q) \\ &\quad + (-1)^{m+1} \frac{2(2m+2)!}{(2\pi)^{2m+2}} \left[\sum_{n=1}^{\infty} \frac{\ln n}{n^{2m+2}} \cos(2\pi n q) - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sin(2\pi n q)}{n^{2m+2}} \right]. \end{aligned} \tag{3.24}$$

Lemma 3.6. *For $0 \leq q \leq 1$ the function $A_2(q)$ is given by*

$$\begin{aligned} A_2(q) &= (1 - \gamma - \ln 2\pi)\left(q^2 - q + \frac{1}{6}\right) \\ &\quad - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \cos(2\pi n q) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n q)}{n^2}. \end{aligned} \tag{3.25}$$

In particular,

$$\begin{aligned} A_2(1) &= 2\zeta'(-1), \\ A_2\left(\frac{1}{2}\right) &= -\zeta'(-1) - \frac{1}{12}\ln 2, \\ A_2\left(\frac{1}{4}\right) &= -\frac{1}{4}\zeta'(-1) + \frac{G}{2\pi}, \end{aligned}$$

where G is Catalan's constant.

Proof: The expression (3.25) follows directly from (3.24) at $m = 0$. □

Special values of the derivative of the Hurwitz zeta function at the rational arguments $q = \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$, for k odd, have been given in [8]. Special values of $A_k(q)$ for $q > 1$ can be obtained by using Lemma 3.3 a sufficient number of times.

Example 3.7.

$$\begin{aligned} A_2(2) &= 2\zeta'(-1), \\ A_2(3) &= 2\zeta'(-1) + 4\ln 2, \\ A_2\left(\frac{3}{2}\right) &= -\zeta'(-1) - \frac{13}{12}\ln 2, \\ A_2\left(\frac{5}{4}\right) &= -\frac{1}{4}\zeta'(-1) + \frac{G}{2\pi} - \ln 2. \end{aligned}$$

Several of the indefinite integrals derived in the next sections can be conveniently expressed in terms of a family of functions closely related to the *negapolygamma* family introduced by Gosper [6] as

$$\begin{aligned}\psi_{-1}(q) &:= \ln\Gamma(q),\\ \psi_{-k}(q) &:= \int_0^q \psi_{-k+1}(t)\,dt, \quad k \geq 2.\end{aligned} \tag{3.26}$$

These functions were later reconsidered by Adamchik [1] in the form

$$\psi_{-k}(q) = \frac{1}{(k-2)!}\int_0^q (q-t)^{k-2}\ln\Gamma(t)\,dt, \quad k \geq 2. \tag{3.27}$$

We introduce the *balanced negapolygamma* functions by

$$\psi^{(-k)}(q) := \frac{1}{k!}\left[A_k(q) - H_{k-1}B_k(q)\right], \tag{3.28}$$

where $k \in \mathbb{N}$ and H_r is the harmonic number ($H_0 := 0$). For instance,

$$\psi^{(-1)}(q) = A_1(q) = \ln\Gamma(q) + \zeta'(0), \tag{3.29}$$

$$\psi^{(-2)}(q) = \frac{1}{2}A_2(q) - \frac{1}{2}B_2(q). \tag{3.30}$$

Note. We can express the derivative of the Hurwitz zeta function at non-positive integers as

$$\zeta'(-r,q) = r!\psi^{(-1-r)}(q) + \frac{H_r}{1+r}B_{1+r}(q), \quad r \in \mathbb{N}_0. \tag{3.31}$$

Lemma 3.8. *The balanced negapolygammas can be expressed as*

$$\psi^{(-k)}(q) = e^{-\gamma z}\frac{\partial}{\partial z}\left[e^{\gamma z}\frac{\zeta(z,q)}{\Gamma(1-z)}\right]\Bigg|_{z=1-k}. \tag{3.32}$$

Proof: Perform the derivative and use $\psi(k) = H_{k-1} - \gamma$, in addition to the definition (3.1) and the identity (1.6). □

We shall now study some of the properties of the balanced negapolygamma functions $\psi^{(-k)}(q)$. The adjective *balanced* is motivated by the following result:

Lemma 3.9. *For* $k \in \mathbb{N}$,

$$\int_0^1 \psi^{(-k)}(q)\,dq = 0. \tag{3.33}$$

Proof: Use (3.6) and the analogous result for the Bernoulli polynomials,

$$\int_0^1 B_k(q)\,dq = 0,$$

for $k \in \mathbb{N}$. □

Lemma 3.10. *For* $k \in \mathbb{N}$,

$$\frac{d}{dq}\psi^{(-k)}(q) = \psi^{(-k+1)}(q). \tag{3.34}$$

Proof: For $k \geq 2$, (3.34) is a direct consequence of Lemma 3.4 and the well-known property of the Bernoulli polynomials,

$$\frac{d}{dq}B_k(q) = kB_{k-1}(q). \tag{3.35}$$

The case $k = 1$ follows directly from (3.29) since

$$\frac{d}{dq}\ln\Gamma(q) = \psi(q) = \psi^{(0)}(q).$$ □

The functions $\psi^{(-k)}(q)$ defined by (3.28) are thus closely related to the ones introduced by Gosper. In fact, the precise relationship between both families of polygammas of negative order is

$$\psi^{(-k)}(q) = \psi_{-k}(q) + \sum_{r=0}^{k-1}\frac{q^{k-1-r}}{r!(k-1-r)!}[\zeta'(-r) + H_r\zeta(-r)], \tag{3.36}$$

where we have used the evaluation

$$\psi^{(-1-r)}(0) = \frac{1}{r!}[\zeta'(-r) + H_r\zeta(-r)], \quad r \in \mathbb{N}. \tag{3.37}$$

Note. According to [1],

$$\zeta'(-r) + H_r\zeta(-r) = -\ln A_r, \tag{3.38}$$

where the A_r are the generalized Glaisher constants defined by Bendersky [3].

Lemma 3.11. *The balanced negapolygamma functions* $\psi^{(-k)}(q)$ *admit the following Fourier expansion in the range* $0 \leq q \leq 1$:

$$\psi^{(-k)}(q) = \frac{2}{(2\pi)^k}\left[\sum_{n=1}^{\infty}\frac{\ln(2\pi n) + \gamma}{n^k}\cos\left(2\pi nq - \frac{k\pi}{2}\right)\right.$$
$$\left. - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^k}\sin\left(2\pi nq - \frac{k\pi}{2}\right)\right]. \tag{3.39}$$

Proof: In the definition (3.28) of $\psi^{(-k)}(q)$ substitute, depending on the parity of k, the Fourier expansions (3.23) or (3.24) for the functions $A_k(q)$ and (3.19) or (3.22) for the Bernoulli polynomials. □

Lemma 3.12. *For $k \in \mathbb{N}$,*

$$\psi^{(-k)}(q+1) = \psi^{(-k)}(q) + \frac{q^{k-1}}{(k-1)!}[\ln q - H_{k-1}]. \tag{3.40}$$

Proof: In the definition (3.28) use (3.3) and the property

$$B_m(q+1) = B_m(q) + mq^{m-1},$$

satisfied by the Bernoulli polynomials. □

Corollary 3.13. *For $r \in \mathbb{N}$,*

$$\psi^{(-1-r)}(1) = \psi^{(-1-r)}(0) = \frac{1}{r!}[\zeta'(-r) + H_r\zeta(-r)]. \tag{3.41}$$

Proof: Set $k = 1 + r$ and evaluate (3.40) at $q = 0$. Then use (3.37). □

4. Indefinite integrals of polygamma functions

Integral formulae involving the polygamma functions can be obtained from the corresponding ones for $\zeta(z, q)$, like (2.3) or (2.5), by taking the limit $z \to m = 2, 3, \ldots$. This limit is not trivial in general, because of the vanishing of the Pochhammer symbol $(1-z)_{j+1}$ for $j > m-2$, and the appearance of the function $\zeta(1, q)$ on the right-hand side of the formulae cited above. Similarly, differentiation at $z = 1 - k$ leads to evaluations involving the functions $A_k(q)$ and the negapolygamma functions. Because of the connection (3.29) between $\psi^{(-1)}(q)$ and $\ln\Gamma(q)$, and of the latter with the function $\ln\sin(\pi q)$, we also obtain indefinite integral formulae involving these last two functions.

The next theorem gives the moments of the polygamma functions in terms of themselves and the balanced negapolygammas defined in Section 3. The proof of the theorem rests on the following result:

Lemma 4.1. *Let $p \in \mathbb{N}_0$. Then*

$$\sum_{r=0}^{p} (-1)^r \binom{p+1}{r+1} \frac{q^{p-r} B_{r+1}(a+bq)}{b^{r+1}} = q^{p+1} + (-1)^p \frac{B_{p+1}(a)}{b^{p+1}}. \tag{4.1}$$

Proof: The basic identity

$$B_{p+1}(x+y) = \sum_{r=0}^{p+1} \binom{p+1}{r} B_r(x) y^{p+1-r} \tag{4.2}$$

appears in [9] 19:5:3. The identity of the lemma is obtained by replacing x by $a + bq$ and y by $-bq$ in (4.2). □

Theorem 4.2. *Let $n \in \mathbb{N}_0$, $m \in \mathbb{N}$. Then*

$$\int q^n \psi^{(m)}(a+bq)\,dq = n! \sum_{j=0}^{n} \frac{(-1)^j}{b^{j+1}(n-j)!} q^{n-j} \psi^{(m-j-1)}(a+bq). \tag{4.3}$$

Proof: In view of relationship (1.10), we set $z = m + \epsilon$ in (2.3) and take the limit $\epsilon \to 0$. For a general value of n there will be three kinds of terms to consider in the sum on the right-hand side of (2.3):

(a) For $0 \leq j \leq m-2$ no singularities will arise and we simply have

$$\lim_{\epsilon \to 0} \frac{\zeta(m-j+\epsilon, q)}{(-m-\epsilon)_{j+1}} = \frac{(-1)^{m+1}}{m!} \psi^{(m-j-1)}(q).$$

(b) For $j = m-1$ we use the results

$$(-m-\epsilon)_m = (-1)^m m! [1 + H_m \epsilon + O(\epsilon^2)]$$

and

$$\zeta(1+\epsilon, q) = \frac{1}{\epsilon} - \psi(q) + O(\epsilon^2),$$

where $\psi(q) = \psi^{(0)}(q)$ is the usual digamma function, to obtain

$$\left. \frac{\zeta(m-j+\epsilon, q)}{(-m-\epsilon)_{j+1}} \right|_{j=m-1} = \frac{(-1)^{m+1}}{m!} \left[-\frac{1}{\epsilon} + \psi(q) + H_m + O(\epsilon) \right].$$

(c) For $j \geq m$, say $j = m + r$ with $r \geq 0$, we use

$$(-m-\epsilon)_{m+1+r} = (-1)^{m+1} m! r! \epsilon [1 + (H_m - H_r)\epsilon + O(\epsilon^2)]$$

and

$$\zeta(-r+\epsilon, q) = \zeta(-r, q) + \epsilon \zeta'(-r, q) + O(\epsilon^2)$$

to obtain

$$\left. \frac{\zeta(m-j+\epsilon, q)}{(-m-\epsilon)_{j+1}} \right|_{j=m+r} = \frac{(-1)^{m+1}}{m!} \left[-\frac{1}{\epsilon} \frac{B_{r+1}(q)}{(r+1)!} + H_m \frac{B_{r+1}(q)}{(r+1)!} \right. \\ \left. + \psi^{(-1-r)}(q) + O(\epsilon) \right],$$

where we have used the definition (3.28) of the balanced negapolygamma function and (1.6). When all the terms are added up, we find, in view of Lemma 4.1, that the coefficients of the $1/\epsilon$ singularity and the term proportional to the harmonic number H_m reduce to a q-independent constant, which can be dropped. This proves the theorem. □

A similar result holds for the moments of the digamma function:

Theorem 4.3. *Let $n \in \mathbb{N}_0$. Then*

$$\int q^n \psi(a+bq)\,dq = n! \sum_{j=0}^{n} \frac{(-1)^j}{b^{j+1}(n-j)!} q^{n-j} \psi^{(-j-1)}(a+bq). \tag{4.4}$$

Proof: Use (1.9) in (2.3) and proceed along the same lines as the proof above. □

Note. Result (4.4) can be identified as an extension of Theorem 4.2 to the case $m = 0$.

Note. Adamchik [1, 2] has provided the alternative representations

$$\begin{aligned}\int_0^z x^n \psi(x)\,dx &= (-1)^n \left(\frac{B_{n+1} H_n}{n+1} - \zeta'(-n)\right)\\ &\quad + \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k} \left(\zeta'(-k,z) - \frac{B_{k+1}(z) H_k}{k+1}\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^z x^n \psi(x)\,dx &= \sum_{k=0}^{n-1} (-1)^k z^{n-k} \binom{n}{k} \left(\zeta'(-k) - \frac{B_{k+1}(z) H_k}{k+1}\right)\\ &\quad - \sum_{k=1}^{n} (-1)^k k! \left\{ {n \atop k} \right\} \log G_{k+1}(z+1) + (-1)^n H_n \frac{B_{n+1} - B_{n+1}(z)}{n+1},\end{aligned}$$

where $\{ {n \atop k} \}$ are the Stirling numbers of the second kind and $G_k(z)$ is the multiple Barnes function. The first representation can be directly obtained from (4.4) at $a = 0$ and $b = 1$, by explicit evaluation of the integral between $q = 0$ and $q = z$, making use of (3.37) for $\psi^{(-j-1)}(0)$ and of the definition (3.28) of the negapolygamma functions.

By differentiating formula (2.3) at $z = 1 - m$ we can arrive at the following result for the moments of the balanced negapolygamma functions, which extends Theorem 4.2 to negative values of m:

Theorem 4.4. *Let $n \in \mathbb{N}_0$, $m \in \mathbb{N}$. Then*

$$\int q^n \psi^{(-m)}(a+bq)\,dq = n! \sum_{j=0}^{n} \frac{(-1)^j}{b^{j+1}(n-j)!} q^{n-j} \psi^{(-m-j-1)}(a+bq). \tag{4.5}$$

Proof: Differentiate (2.3) at $z = 1 - m$ and use the result

$$\frac{d}{dz}(1-z)_{j+1}\bigg|_{z=1-m} = -(m)_{j+1}[H_{m+j} - H_{m-1}]$$

to obtain the following result for the moments of the function $A_m(q)$:

$$\int q^n A_m(a+bq)\,dq = m!\,n!\sum_{j=0}^{n} \frac{(-1)^j}{b^{j+1}(n-j)!(m+j+1)!} q^{n-j}[A_{m+j+1}(a+bq) \\ - (H_{m+j} - H_{m-1})B_{m+j+1}(a+bq)]. \tag{4.6}$$

This result is equivalent to the statement of the theorem in view of definition (3.28) of the balanced negapolygammas and Formula (2.7) for the moments of the Bernoulli polynomials. □

As a particular case of the last theorem we obtain a formula for the moments of the loggamma function $\ln\Gamma(q)$.

Example 4.5. Let $n \in \mathbb{N}_0$ and $a,\ b \in \mathbb{R}$. Then

$$\int q^n \ln\Gamma(a+bq)\,dq = \ln\sqrt{2\pi}\,\frac{q^{n+1}}{n+1} + n!\sum_{j=0}^{n}\frac{(-1)^j q^{n-j}}{b^{j+1}(n-j)!}\psi^{(-2-j)}(a+bq). \tag{4.7}$$

This generalizes Gosper's result [6], which establishes that all integrals of the form $\int q^n \ln q!\,dq$ are expressible in terms of $\zeta'(-j,q)$, with $1 \le j \le n+1$.

As particular cases of (4.7) we have

$$\int q^n \ln\Gamma(q)\,dq = \ln\sqrt{2\pi}\,\frac{q^{n+1}}{n+1} + n!\sum_{j=0}^{n}\frac{(-1)^j q^{n-j}}{(n-j)!}\psi^{(-2-j)}(q) \tag{4.8}$$

and

$$\int q^n \ln\Gamma(1-q)\,dq = \ln\sqrt{2\pi}\,\frac{q^{n+1}}{n+1} - n!\sum_{j=0}^{n}\frac{q^{n-j}}{(n-j)!}\psi^{(-2-j)}(1-q). \tag{4.9}$$

Proof: Apply theorem (4.4) to $\psi^{(-1)}(q) = \ln\Gamma(q) - \ln\sqrt{2\pi}$. The expressions (4.8) and (4.9) correspond to $a = 0,\ b = 1$ and $a = 1,\ b = -1$ respectively. □

The two special cases of Example 4.5 are now combined with the reflection formula for the gamma function

$$\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin\pi q} \tag{4.10}$$

to obtain an expression for the moments of $\ln\sin\pi q$.

Example 4.6. Let $n \in \mathbb{N}_0$. Then

$$\int q^n \ln \sin \pi q \, dq = -\frac{q^{n+1} \ln 2}{n+1} - n! \sum_{j=0}^{n} \frac{q^{n-j}}{(j+2)!(n-j)!} \times [(-1)^j A_{j+2}(q) - A_{j+2}(1-q)]. \tag{4.11}$$

Proof: Use the reflection formula for $\Gamma(q)$, results (4.8, 4.9) and the definition (3.28) to produce (4.11). The term that corresponds to the Bernoulli polynomials disappears in view of

$$(-1)^j B_{j+2}(q) = B_{j+2}(1-q). \tag{4.12}$$

□

Example 4.7.

$$\int e^q \ln \sin \pi q \, dq = -e^q \left[\ln 2 + \sum_{j=0}^{\infty} \frac{(-1)^j A_{j+2}(q) - A_{j+2}(1-q)}{(j+2)!} \right]. \tag{4.13}$$

Proof: Divide (4.11) by $n!$ and sum over n. □

Example 4.8. Integrating (4.13) by parts yields

$$\int e^q \operatorname{cotg} \pi q \, dq = \frac{e^q}{\pi} \left[\ln \sin \pi q + \ln 2 + \sum_{j=0}^{\infty} \frac{(-1)^j A_{j+2}(q) - A_{j+2}(1-q)}{(j+2)!} \right]. \tag{4.14}$$

5. Some definite integrals

Some of the definite integral formulae given in [5], in the range (0, 1), can be obtained directly from the indefinite integral formulae given in Sections 2 and 4.

Example 5.1. Evaluate Eq. (2.3) between 0 and 1 to obtain

$$\int_0^1 q^n \zeta(z, a+bq)\, dq = n! \sum_{j=0}^{n-1} \frac{(-1)^j \zeta(z-j-1, a+b)}{b^{j+1}(1-z)_{j+1}(n-j)!} + \frac{n!(-1)^n}{b^{n+1}(1-z)_{n+1}} \left(\zeta(z-n-1, a+b) - \zeta(z-n-1, a)\right). \tag{5.1}$$

As a particular case we obtain formula (12.2) of [5]:

Corollary 5.2. *Let $n \in \mathbb{N}_0$ and $z \in \mathbb{R}$, with $z - n - 1 < 0$. Then*

$$\int_0^1 q^n \zeta(z, q)\, dq = n! \sum_{j=0}^{n-1} \frac{(-1)^j \zeta(z - j - 1)}{(1 - z)_{j+1} (n - j)!}. \tag{5.2}$$

Proof: Set $b = 1$ in (5.1) and use the identity (3.4) and the hypothesis $z - n - 1 < 0$ to get rid of the last term in (5.1) in the limit $a \to 0$. □

The evaluation of formula (4.11) between $q = 0$ and $q = 1$ leads to formula (5.6) of [5]:

Example 5.3. Let $n \in \mathbb{N}_0$. Then

$$\int_0^1 q^n \ln(\sin \pi q)\, dq = -\frac{\ln 2}{n+1} + n! \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \zeta(2k+1)}{(2\pi)^{2k} (n + 1 - 2k)!}. \tag{5.3}$$

Proof: Direct evaluation of the right-hand side of (4.11) gives, in view of property (3.5),

$$\int_0^1 q^n \ln \sin \pi q \; dq = -\frac{\ln 2}{n+1} - n! \sum_{j=0}^{n-1} \frac{1}{(j+1)!(n-j)!} [(-1)^j - 1] \zeta'(-j-1). \tag{5.4}$$

Clearly only the terms with j odd, say $j = 2k - 1$, survive in the sum. (5.3) then follows directly from (3.9). □

Example 5.4. Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned} \int_0^{1/2} q^n \ln(\sin \pi q)\, dq = &-\frac{1}{2^{n+1}} \left[\frac{\ln 2}{n+1} + n! \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(-1)^k (2^{2k} - 1) \zeta(2k+1)}{(2\pi)^{2k} (n + 1 - 2k)!} \right] \\ &- \frac{1 - (-1)^n}{n+1} \zeta'(-n-1). \end{aligned} \tag{5.5}$$

Proof: Use (3.11) in (4.11). □

For instance,

$$\begin{aligned} \int_0^{1/2} q \ln(\sin \pi q)\, dq &= -\frac{1}{8} \ln 2 + \frac{7\zeta(3)}{16\pi^2}, \\ \int_0^{1/2} q^2 \ln(\sin \pi q)\, dq &= -\frac{1}{24} \ln 2 + \frac{3\zeta(3)}{16\pi^2}, \\ \int_0^{1/2} q^3 \ln(\sin \pi q)\, dq &= -\frac{1}{64} \ln 2 + \frac{9\zeta(3)}{64\pi^2} - \frac{93\zeta(5)}{128\pi^4}. \end{aligned}$$

Example 5.5. Formulae (4.7) or (4.8) allow us to derive Gosper's formulae for integrals of $\ln \Gamma(q)$ [6] in a very economical way. For instance, setting $n = 0$, $a = b = 1$ in (4.7) yields

$$\int_0^q \ln\Gamma(q+1)\,dq = q\ln\sqrt{2\pi} + \frac{1}{2}A_2(q+1) - \frac{1}{2}B_2(q+1) - \zeta'(-1) + \frac{1}{2}B_2. \quad (5.6)$$

Evaluation of the right-hand side at $q = \frac{1}{2}$ and $\frac{1}{4}$ yields, respectively,

$$\int_0^{1/2} \ln\Gamma(q+1)\,dq = -\frac{3}{8} - \frac{13}{24}\ln 2 + \frac{1}{2}\ln\sqrt{2\pi} - \frac{3}{2}\zeta'(-1),$$
$$\int_0^{1/4} \ln\Gamma(q+1)\,dq = -\frac{5}{32} - \frac{1}{2}\ln 2 + \frac{1}{4}\ln\sqrt{2\pi} - \frac{9}{8}\zeta'(-1) + \frac{G}{4\pi},$$

which can easily be seen to be equivalent to Gosper's formulae, after using Riemann's functional equation for the Riemann zeta function to express $\zeta'(-1)$ in the form

$$\zeta'(-1) = \frac{\zeta'(2)}{2\pi^2} - \frac{1}{12}(2\ln\sqrt{2\pi} + \gamma - 1). \quad (5.7)$$

Example 5.6. For $k, k' \in \mathbb{N}$,

$$\int_0^1 \psi^{(-k)}(q)\psi^{(-k')}(q)\,dq = \frac{2\cos(k-k')\frac{\pi}{2}}{(2\pi)^{k+k'}}\Bigg[\zeta''(k+k') - 2(\gamma+\ln 2\pi)\zeta'(k+k') + \left\{(\gamma+\ln 2\pi)^2 + \frac{\pi^2}{4}\right\}\zeta(k+k')\Bigg].$$

The special case $k = k' = 1$ reduces to

$$\int_0^1 (\ln\Gamma(q))^2\,dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3}\gamma\ln\sqrt{2\pi} + \frac{4}{3}\ln^2\sqrt{2\pi} - (\gamma + 2\ln\sqrt{2\pi})\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2},$$

given in [5].

Proof: Use the representation (3.32) of the balanced negapolygammas to obtain the desired result, by direct differentiation of the formula

$$\int_0^1 \zeta(z',q)\zeta(z,q)\,dq = \frac{2\Gamma(1-z)\Gamma(1-z')}{(2\pi)^{2-z-z'}}\zeta(2-z-z')\cos\left(\frac{\pi(z-z')}{2}\right), \quad (5.8)$$

valid for real $z,\ z' \le 0$, given in [5]. □

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