THE STORY OF LANDEN, THE HYPERBOLA AND THE ELLIPSE

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Abstract. We establish a relation among the arc lengths of a hyperbola, a circle and an ellipse.

1. Introduction

The problem of rectification of conics was a central question of analysis in the 18^{th} century. The goal of this note is to describe Landen's work on rectifying the arc of a hyperbola in terms of an ellipse and a circle. Naturally, Landen's language is that of his time, in terms of *fluents* and *fluxions*, and his arguments are not rigorous in the modern sense.

The main result presented here is a special relation between the length of an ellipse, the length of a hyperbolic segment, and the length of circle. The proof is based on a generalization of Euler's formula for the lemniscatic curve as described in [4].

2. The hyperbola

The arc length of the equilateral hyperbola

$$h(t) = \sqrt{t^2-1}, \quad t \geq 1 \tag{2.1}$$

starting at $t=1$ is given by

$$L_h(x) = \int_1^x \sqrt{\frac{2t^2-1}{t^2-1}}\, dt \tag{2.2}$$

as a function of the terminal point $t = x$. The tangent line to the hyperbola at $t = x$ is

$$T_h(t) \quad = \quad \sqrt{x^2-1} + \frac{x}{\sqrt{x^2-1}}(t-x), \tag{2.3}$$

whose intersection with the t-axis is $t = 1/x \in (0,1)$. The line

$$N_h(t) = -\frac{\sqrt{x^2-1}}{x} t \tag{2.4}$$

is the perpendicular to L_h passing through the origin. The lines T_h and L_h intersect at the point

$$P_h \quad = \quad \left(\frac{x}{2x^2-1}, -\frac{\sqrt{x^2-1}}{2x^2-1} \right). \tag{2.5}$$

Date: April 27, 2001.
1991 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Landen transformations.

The distance from $(x, h(x))$ to the common point P_h is

$$g_h(x) = 2x\sqrt{\frac{x^2-1}{2x^2-1}}. \tag{2.6}$$

It was observed by Maclaurin, D'Alambert, and Landen that

$$f_h(x) := g_h(x) - L_h(x) = 2x\sqrt{\frac{x^2-1}{2x^2-1}} - \int_1^x \sqrt{\frac{2t^2-1}{t^2-1}}dt \tag{2.7}$$

is easier to analyze than the arc length $L_h(x)$.

Proposition 2.1. Let

$$F_h(z) = \frac{1}{2}\int_z^1 \sqrt{\frac{t}{1-t^2}}\,dt. \tag{2.8}$$

Then

$$F_h(z) = f_h(x), \tag{2.9}$$

where

$$z = \frac{1}{2x^2-1}. \tag{2.10}$$

Proof. Make the change of variable (2.10) in (2.7). Then $f_h(x)$ becomes

$$F_h(z) = \sqrt{\frac{1-z^2}{z}} + \frac{1}{2}\int_1^z \frac{ds}{s^{3/2}\sqrt{1-s^2}}. \tag{2.11}$$

in terms of the new variable $z = 1/(2x^2-1)$. Since

$$\frac{d}{ds}\sqrt{\frac{1-s^2}{s}} = \frac{-1-s^2}{2s^{3/2}\sqrt{1-s^2}},$$

integrating from 1 to z reduces (2.11) to (2.8). ☐

3. The ellipse

The equation of the ellipse can be written as

$$e(t) = \sqrt{2(1-t^2)}, \quad |t| \leq 1. \tag{3.1}$$

In this case the tangent line at $t = r$ is

$$T_e(t) = \sqrt{2(1-r^2)} - \sqrt{\frac{2r^2}{1-r^2}}\,(t-r),$$

and the line

$$N_e(t) = \sqrt{\frac{1-r^2}{2r^2}}\,t$$

is the perpendicular to T_e through the origin. These two lines intersect at the point

$$P_e = \left(\frac{2r}{1+r^2}, \frac{\sqrt{r(1-r^2)}}{1+r^2}\right), \tag{3.2}$$

and the distance from $(r, e(r))$ to the common point P_e is

$$g_e(r) = r\sqrt{\frac{1-r^2}{1+r^2}}. \tag{3.3}$$

We express the function g_e in terms of the new variable $z = r^2$ as

$$g_e(z) \quad = \quad \sqrt{\frac{z(1-z)}{1+z}}. \tag{3.4}$$

4. The connection

We now evaluate the function $F_h(z)$ in (2.8) at two points y, $z \in (0,1)$ related via the bilinear transformation $z = (1-y)/(1+y)$. We have

$$F_h(z) + F_h(y) \quad = \quad \frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\, ds + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\, ds.$$

The change of variable $\sigma = (1-s)/(1+s)$ in the second integral yields

$$F_h(z) + F_h(y) \quad = \quad \frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\, ds + \frac{1}{2}\int_0^y \frac{\sqrt{1-\sigma}}{(1+\sigma)^{3/2}\sqrt{\sigma}}\, d\sigma.$$

Now recall the function $g_e(z)$ in (3.4) and its differential

$$\frac{dg_e}{dz} \quad = \quad \frac{1}{2}\frac{\sqrt{1-z}}{\sqrt{z}(1+z)^{3/2}} - \frac{1}{2}\frac{\sqrt{z}}{\sqrt{1-z^2}}.$$

Therefore

$$F_h(z) + F_h(y) \quad = \quad g_e(z) - g_e(1) + \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\, dt.$$

Now observe that $g_e(1) = 0$ and introduce the absolute constant

$$L \quad := \quad \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\, dt \tag{4.1}$$

so that

$$F_h(z) + F_h(y) \quad = \quad g_e(z) + L. \tag{4.2}$$

Thus we have established the following integral relation.

Theorem 4.1. Let $y \in (0,1)$ and $z = (1-y)/(1+y)$. Then

$$\frac{1}{2}\int_y^1 \sqrt{\frac{s}{1-s^2}}\, ds + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\, ds \quad = \quad \sqrt{\frac{z(1-z)}{1+z}} + L \tag{4.3}$$

with the absolute constant L in (4.1).

Proof. Let

$$\begin{aligned} G_h(z) \quad &= \quad F_h(z) + F_h(y) \\ &= \quad \frac{1}{2}\int_{(1-z)/(1+z)}^1 \sqrt{\frac{s}{1-s^2}}\, ds + \frac{1}{2}\int_z^1 \sqrt{\frac{s}{1-s^2}}\, ds, \end{aligned}$$

so that

$$\frac{dG_h(z)}{dz} \quad = \quad \frac{1}{2}\frac{\sqrt{1-z}}{\sqrt{z}(1+z)^{3/2}} - \frac{1}{2}\frac{\sqrt{z}}{\sqrt{1-z^2}}. \tag{4.4}$$

Integrating (4.4) gives

$$G_h(z) \quad = \quad \sqrt{\frac{z(1-z)}{1+z}} + L \tag{4.5}$$

By letting $z=0$, the constant L is easily evaluated as

$$L := \frac{1}{2}\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta = \frac{\pi\sqrt{2\pi}}{\Gamma^2(1/4)} \tag{4.6}$$

using Wallis' formula. □

We now follow Landen to establish the value of L in terms of elliptic arcs.

The equation (4.2) simplifies if we evaluate it at the fixed point $z^*=\sqrt{2}-1$ of the transformation $z=(1-y)/(1+y)$. In terms of the x variable, the fixed point is

$$x^* = \sqrt{1+\frac{1}{\sqrt{2}}} = \sqrt{2}\cos(\pi/8). \tag{4.7}$$

Indeed

$$F_h(z^*) = \frac{1}{2}(\sqrt{2}-1+L). \tag{4.8}$$

Now introduce the complementary integral

$$M := \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t(1-t^2)}}, \tag{4.9}$$

and observe that

$$L+M = L_e(1) = \frac{1}{2}\int_0^1 \sqrt{\frac{1+t}{t(1-t)}}\,dt$$

where $L_e(1)$ is a quarter of the length of the ellipse.

Theorem 4.2. The integrals L and M satisfy

$$\begin{aligned} L+M &= L_e(1)\\ L\times M &= \frac{\pi}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} L &= \frac{1}{2}\left(L_e(1)-\sqrt{L_e(1)^2-\pi}\right)\\ M &= \frac{1}{2}\left(L_e(1)+\sqrt{L_e(1)^2-\pi}\right). \end{aligned}$$

Proof. Observe that for $q\in\mathbb{Q}$ we have

$$\frac{d(t^q\sqrt{1-t^2})}{dt} = \frac{qt^{q-1}-(q+1)\,t^{q+1}}{\sqrt{1-t^2}} \tag{4.10}$$

and integrating from 0 to 1 we obtain

$$\int_0^1 \frac{t^{q-1}}{\sqrt{1-t^2}}\,dt = \frac{q+1}{q}\int_0^1 \frac{t^{q+1}}{\sqrt{1-t^2}}\,dt. \tag{4.11}$$

For example, with $q = 3/2$ it yields

$$2L = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t^2}}\, dt \quad = \quad \frac{5}{3}\int_0^1 \frac{t^{5/2}}{\sqrt{1-t^2}}\, dt$$

Iteration of this recurrence yields, after m steps,

$$2L \quad = \quad \prod_{j=1}^{2m+1} (2j-1)^{(-1)^{j+1}} \int_0^1 \frac{t^{2m+1/2}}{\sqrt{1-t^2}} dt. \tag{4.12}$$

Similarly, starting with $q = 1/2$ we get after m steps

$$2M \quad = \quad \prod_{j=1}^{2m} (2j-1)^{(-1)^{j}} \int_0^1 \frac{t^{2m-1/2}}{\sqrt{1-t^2}} dt. \tag{4.13}$$

Iteration of (4.10) with initial values $q = 0$ and $q = 1$ yields for

$$A \quad := \quad \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}$$

and

$$B \quad := \quad \int_0^1 \frac{t\, dt}{\sqrt{1-t^2}} = 1,$$

the expressions

$$\begin{aligned} A \quad &= \quad \prod_{j=1}^{2m} j^{(-1)^j} \int_0^1 \frac{t^{2m}\, dt}{\sqrt{1-t^2}} \\ B \quad &= \quad \prod_{j=1}^{2m} (j+1)^{(-1)^j} \int_0^1 \frac{t^{2m+1}\, dt}{\sqrt{1-t^2}} \end{aligned}$$

and

$$\frac{2L}{A} \quad = \quad \prod_{j=1}^{2m} (2j-1)^{(-1)^{j+1}} j^{(-1)^{j+1}} \frac{\int_0^1 t^{2m+1/2}(1-t^2)^{-1/2}\, dt}{\int_0^1 t^{2m}(1-t^2)^{-1/2}\, dt} \times (4m+1).$$

and

$$\frac{2M}{B} \quad = \quad \prod_{j=1}^{2m} (2j-1)^{(-1)^{j}} j^{(-1)^{j}} \frac{\int_0^1 t^{2m-1/2}(1-t^2)^{-1/2}\, dt}{\int_0^1 t^{2m+1}(1-t^2)^{-1/2}\, dt} \times \frac{1}{2m+1}.$$

As $m \to \infty$, the quotient of the integrals converges to 1 and we obtain

$$2L \times 2M \quad = \quad \frac{\pi}{2} \lim_{m\to\infty} \frac{4m+1}{2m+1} = \pi. \tag{4.14}$$

□

We now write $\pi/2 = L_c(1)$ as a quarter of the length of the circle in analogy to $L_e(1)$.

Theorem 4.3. The length of the hyperbolic segment is given by

$$L_h\left(\sqrt{\frac{1}{2-\sqrt{2}}}\right) \quad = \quad \frac{\sqrt{2}+1}{2} - \frac{1}{4}\sqrt{(L_e(1)^2 - 4L_c(1))} - L_e(1). \tag{4.15}$$

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