THE p-ADIC VALUATION OF k-CENTRAL BINOMIAL COEFFICIENTS

ARMIN STRAUB, VICTOR H. MOLL, AND TEWODROS AMDEBERHAN

ABSTRACT. The coefficients $c(n,k)$ defined by

$$(1-k^2x)^{-1/k} = \sum_{n\geq 0} c(n,k)x^n$$

reduce to the central binomial coefficients $\binom{2n}{n}$ for $k=2$. Motivated by a question of H. Montgomery and H. Shapiro for the case $k=3$, we prove that $c(n,k)$ are integers and study their divisibility properties.

1. INTRODUCTION

In a recent issue of the American Mathematical Monthly, Hugh Montgomery and Harold S. Shapiro proposed the following problem (Problem 11380, August-September 2008):

For $x \in \mathbb{R}$, let

$$\binom{x}{n} = \frac{1}{n!}\prod_{j=0}^{n-1}(x-j). \tag{1.1}$$

For $n \geq 1$, let a_n be the numerator and q_n the denominator of the rational number $\binom{-1/3}{n}$ expressed as a reduced fraction, with $q_n > 0$.

(1) Show that q_n is a power of 3.
(2) Show that a_n is odd if and only if n is a sum of distinct powers of 4.

Our approach to this problem employs Legendre's remarkable expression [7]:

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{1.2}$$

that relates the p-adic valuation of factorials to the sum of digits of n in base p. For $m \in \mathbb{N}$ and a prime p, the p-adic valuation of m, denoted by $\nu_p(m)$, is the highest power of p that divides m. The expansion of $m \in \mathbb{N}$ in base p is written as

$$m = a_0 + a_1p + \cdots + a_dp^d, \tag{1.3}$$

Date: September 30, 2009.
2000 *Mathematics Subject Classification.* Primary 11A51, Secondary 11A63.

with integers $0 \leq a_j \leq p-1$ and $a_d \neq 0$. The function s_p in (1.2) is defined by

$$s_p(m) := a_0 + a_1 + \cdots + a_d. \tag{1.4}$$

Since, for $n > 1$, $\nu_p(n) = \nu_p(n!) - \nu_p((n-1)!)$, it follows from (1.2) that

$$\nu_p(n) = \frac{1 + s_p(n-1) - s_p(n)}{p-1}. \tag{1.5}$$

The p-adic valuations of binomial coefficients can be expressed in terms of the function s_p:

$$\nu_p\left(\binom{n}{k}\right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}. \tag{1.6}$$

In particular, for the central binomial coefficients $C_n := \binom{2n}{n}$ and $p = 2$, we have

$$\nu_2\left(C_n\right) = 2s_2(n) - s_2(2n) = s_2(n). \tag{1.7}$$

Therefore, C_n is always even and $\frac{1}{2}C_n$ is odd precisely whenever n is a power of 2. This is a well-known result.

The central binomial coefficients C_n have the generating function

$$(1-4x)^{-1/2} = \sum_{n \geq 0} C_n x^n. \tag{1.8}$$

The binomial theorem shows that the numbers in the Montgomery-Shapiro problem bear a similar generating function

$$(1-9x)^{-1/3} = \sum_{n \geq 0} \binom{-\frac{1}{3}}{n} (-9x)^n. \tag{1.9}$$

It is natural to consider the coeffients $c(n,k)$ defined by

$$(1-k^2x)^{-1/k} = \sum_{n \geq 0} c(n,k) x^n, \tag{1.10}$$

which include the central binomial coefficients as a special case. We call $c(n,k)$ the *k-central binomial coefficients*. The expression

$$c(n,k) = (-1)^n \binom{-\frac{1}{k}}{n} k^{2n} \tag{1.11}$$

comes directly from the binomial theorem. Thus, the Montgomery-Shapiro question from (1.1) deals with arithmetic properties of

$$\binom{-\frac{1}{3}}{n} = (-1)^n \frac{c(n,3)}{3^{2n}}. \tag{1.12}$$

2. The integrality of $c(n,k)$

It is a simple matter to verify that the coefficients $c(n,k)$ are rational numbers. The expression produced in the next proposition is then employed to prove that $c(n,k)$ are actually integers. The next section will explore divisibility properties of the integers $c(n,k)$.

Proposition 2.1. *The coefficient* $c(n,k)$ *is given by*

$$c(n,k) = \frac{k^n}{n!}\prod_{m=1}^{n-1}(1+km). \tag{2.1}$$

Proof. The binomial theorem yields

$$\begin{aligned}(1-k^2x)^{-1/k} &= \sum_{n\geq 0}\binom{-\frac{1}{k}}{n}(-k^2x)^n \\ &= \sum_{n\geq 0}\frac{k^n}{n!}\left(\prod_{m=1}^{n-1}(1+km)\right)x^n,\end{aligned}$$

and (2.1) has been established. □

An alternative proof of the previous result is obtained from the simple recurrence

$$c(n+1,k) = \frac{k(1+kn)}{n+1}c(n,k), \qquad \text{for } n\geq 0, \tag{2.2}$$

and its initial condition $c(0,k)=1$. To prove (2.2), simply differentiate (1.10) to produce

$$k(1-k^2x)^{-1/k-1} = \sum_{n\geq 0}(n+1)c(n+1,k)x^n \tag{2.3}$$

and multiply both sides by $1-k^2x$ to get the result.

Note. The coefficients $c(n,k)$ can be written in terms of the Beta function as

$$c(n,k) = \frac{k^{2n}}{nB(n,1/k)}. \tag{2.4}$$

This expression follows directly by writing the product in (2.1) in terms of the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$ and the identity

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.5}$$

The proof employs only the most elementary properties of the Euler's Gamma and Beta functions. The reader can find details in [1]. The conclusion is that we have an integral expression for $c(n,k)$, given by

$$c(n,k)\int_0^1 (1-u^{1/n})^{1/k-1}\,du = k^{2n}. \tag{2.6}$$

It is unclear how to use it to further investigate $c(n,k)$.

In the case $k=2$, we have that $c(n,2)=C_n$ is a positive integer. This result extends to all values of k.

Theorem 2.2. *The coefficient $c(n,k)$ is a positive integer.*

Proof. First observe that if p is a prime dividing k, then the product in (2.1) is relatively prime to p. Therefore we need to check that $\nu_p(n!) \leq \nu_p(k^n)$. This is simple:

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1} \leq n \leq \nu_p(k^n). \tag{2.7}$$

Now let p be a prime not dividing k. Clearly,

$$\nu_p(c(n,k)) = \nu_p\left(\prod_{m<n}(1+km)\right) - \nu_p\left(\prod_{m<n}(1+m)\right). \tag{2.8}$$

To prove that $c(n,k)$ is an integer, we compare the p-adic valuations appearing in (2.8). Observe that $1+m$ is divisible by p^α if and only if m is of the form $\lambda p^\alpha - 1$. On the other hand, $1+km$ is divisible by p^α precisely when m is of the form $\lambda p^\alpha - i_{p^\alpha}(k)$, where $i_{p^\alpha}(k)$ denotes the inverse of k modulo p^α in the range $1, 2, \cdots, p^\alpha - 1$. Thus,

$$\nu_p(c(n,k)) = \sum_{\alpha \geq 1} \left\lfloor \tfrac{n + i_{p^\alpha}(k) - 1}{p^\alpha} \right\rfloor - \left\lfloor \tfrac{n}{p^\alpha} \right\rfloor. \tag{2.9}$$

The claim now follows from $i_{p^\alpha}(k) \geq 1$. ☐

Next, Theorem 2.2 will be slightly strengthened and an alternative proof be provided.

Theorem 2.3. *For $n>0$, the coefficient $c(n,k)$ is a positive integer divisible by k.*

Proof. Expanding the right hand side of the identity

$$(1-k^2x)^{-1} = \left((1-k^2x)^{-1/k}\right)^k \tag{2.10}$$

by the Cauchy product formula gives

$$\sum_{i_1+\cdots+i_k=m} c(i_1,k)c(i_2,k)\cdots c(i_k,k) = k^{2m}, \tag{2.11}$$

where the multisum runs through all the k-tuples of non-negative integers. Obviously $c(0,k)=1$ and it is easy to check that $c(1,k)=k$. We proceed by induction on n, so we assume the assertion is valid for $c(1,k), c(2,k), \cdots, c(n-1,k)$. We prove the same is true for $c(n,k)$. To this end, break up (2.11) as

$$kc(n,k) + \sum_{i_1+\cdots+i_k=n} c(i_1,k)c(i_2,k)\cdots c(i_k,k) = k^{2n}. \tag{2.12}$$

Hence by the induction assumption $kc(n,k)$ is an integer.

To complete the proof, divide (2.12) through by k^2 and rewrite as follows

$$\frac{c(n,k)}{k} = k^{2n-2} - \frac{1}{k^2} \sum_{\substack{i_1+\cdots+i_k=n \\ 0 \leq i_j < n}} c(i_1,k)c(i_2,k)\cdots c(i_k,k). \tag{2.13}$$

The key point is that each summand in (2.13) contains *at least two* terms, each one divisible by k. □

Note. W. Lang [6] has studied the numbers appearing in the generating function

$$c2(l;x) := \frac{1-(1-l^2x)^{1/l}}{lx}, \tag{2.14}$$

that bears close relation to the case $k = -l < 0$ of equation (1.10). The special case $l = 2$ yields the Catalan numbers. The author establishes the integrality of the coefficients in the expansion of $c2$ and other related functions.

3. The valuation of $c(n,k)$

We consider now the p-adic valuation of $c(n,k)$. The special case when p divides k is easy, so we deal with it first.

Proposition 3.1. *Let p be a prime that divides k. Then*

$$\nu_p(c(n,k)) = \nu_p(k)n - \frac{n - s_p(n)}{p-1}. \tag{3.1}$$

Proof. The p-adic valuation of $c(n,k)$ is given by

$$\nu_p(c(n,k)) = \nu_p(k)n - \nu_p(n!) = \nu_p(k)n - \frac{n-s_p(n)}{p-1}. \tag{3.2}$$

Finally note that $s_p(n) = O(\log n)$. □

Note. For p, $k \neq 2$, we have $\nu_p(c(n,k)) \sim \left(\nu_p(k) - \frac{1}{p-1}\right) n$, as $n \to \infty$.

We now turn attention to the case where p does not divide k. Under this assumption, the congruence $kx \equiv 1 \bmod p^\alpha$ has a solution. Elementary arguments of p-adic analysis can be used to produce a p-adic integer that yields the inverse of k. This construction proceeds as follows: first choose b_0 in the range $\{1, 2, \cdots, p-1\}$ to satisfy $kb_0 \equiv 1 \bmod p$. Next, choose c_1, satisfying $kc_1 \equiv 1 \bmod p^2$ and write it as $c_1 = b_0 + pb_1$ with $0 \leq b_1 \leq p-1$. Proceeding in this manner, we obtain a sequence of integers $\{b_j : j \geq 0\}$, such that $0 \leq b_j \leq p-1$ and the partial sums of the *formal object* $x = b_0 + b_1p + b_2p^2 + \cdots$ satisfy

$$k(b_0 + b_1p + \cdots + b_{j-1}p^{j-1}) \equiv 1 \bmod p^j \tag{3.3}$$

This is the standard definition of a p-adic integer and

$$i_{p^\infty}(k) = \sum_{j=0}^{\infty} b_j p^j \tag{3.4}$$

is the inverse of k in the ring of p-adic integers. The reader will find in [3] and [8] information about this topic.

Note. It is convenient to modify the notation in (3.4) and write it as

$$i_{p^\infty}(k) = 1 + \sum_{j=0}^{\infty} b_j p^j \tag{3.5}$$

where $0 \leq b_j < p$. This is always possible since the first coefficient cannot be zero. Then, b_0 is defined by $k(1+b_0) \equiv 1 \bmod p$, or, equivalenty, $k(1+b_0) = 1 + \lambda_0 p$ for some $0 \leq \lambda_0 < k$. Therefore, $b_0 = (1+\lambda_0 p)/k - 1 = \lfloor \lambda_0 p/k \rfloor$. Likewise, for every $j \geq 1$ we have $k(1 + b_0 + b_1 p + \cdots + b_j p^j) = 1 + \lambda_j p^{j+1}$ for some $0 \leq \lambda_j < k$. By induction this reduces to $1 + \lambda_{j-1} p^j + k b_j p^j = 1 + \lambda_j p^{j+1}$, or, eqivalently,

$$b_j = \frac{\lambda_j p - \lambda_{j-1}}{k} = \lfloor \lambda_j p/k \rfloor. \tag{3.6}$$

Therefore it has been shown that the coefficients b_j only take values amongst $\lfloor p/k \rfloor, \lfloor 2p/k \rfloor, \ldots, \lfloor (k-1)p/k \rfloor$. Furthermore observe that $0 \leq \lambda_j < k$ is the solution to

$$\lambda_j \equiv -p^{-1-j} \bmod k. \tag{3.7}$$

It follows that the b_j are periodic with period the multiplicative order of p in $\mathbb{Z}/k\mathbb{Z}$.

The analysis of $\nu_p(c(n,k))$ for those primes p not dividing k begins with a characterization of those indices for which $\nu_p(c(n,k)) = 0$, that is, p does not divide $c(n,k)$. The result is expressed in terms of the expansions of n in base p, written as

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_d p^d, \tag{3.8}$$

and the p-adic expansion of the inverse of k as given by (3.5).

Theorem 3.2. *Let p be a prime that does not divide k. Then $\nu_p(c(n,k)) = 0$ if and only if $a_j + b_j < p$ for all j in the range $1 \leq j \leq d$.*

Proof. It follows from (2.9) that $c(n,k)$ is not divisible by p precisely when

$$\left\lfloor \frac{1}{p}\left(n + \sum_j b_j p^j\right) \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor, \tag{3.9}$$

for all $\alpha \geq 1$, or equivalently, if and only if

$$\sum_{j=0}^{\alpha-1} (a_j + b_j)p^j < p^\alpha, \tag{3.10}$$

for all $\alpha \geq 1$. An inductive argument shows that this is equivalent to the condition $a_j + b_j < p$ for all j. Naturally, the a_j vanish for $j > d$, so it is sufficient to check $a_j + b_j < p$ for all $j \leq d$. □

Corollary 3.3. *For all primes $p > k$ and $d \in \mathbb{N}$, we have $\nu_p(c(p^d, k)) = 0$.*

Proof. The coefficients of $n = p^d$ in Theorem 3.2 are $a_j = 0$ for $0 \leq j \leq d-1$ and $a_d = 1$. Therefore the restrictions on the coefficients b_j become $b_j < p$ for $0 \leq j \leq d-1$ and $b_d < p-1$. It turns out that $b_j \neq p-1$ for all $j \in \mathbb{N}$. Otherwise, for some $r \in \mathbb{N}$, we have $b_r = p-1$ and the equation

$$k\left(1 + \sum_{j=0}^{r-1} b_j p^j + b_r p^r\right) \equiv k\left(1 + \sum_{j=0}^{r-1} b_j p^j - p^r\right) \equiv 1 \bmod p^{r+1}, \tag{3.11}$$

is impossible in view of

$$-kp^r < k\left(1 + \sum_{j=0}^{r-1} b_j p^j - p^r\right) < 0. \tag{3.12}$$

□

Now we return again to the Montgomery-Shapiro question. The identity (1.12) shows that the denominator q_n is a power of 3. We now consider the indices n for which $c(n,3)$ is odd and provide a proof of the second part of their problem.

Corollary 3.4. *The coefficient $c(n,3)$ is odd precisely when n is a sum of distinct powers of 4.*

Proof. The result follows from Theorem 3.2 and the explicit formula

$$i_{2^\infty}(3) = 1 + \sum_{j=0}^{\infty} 2^{2j+1}, \tag{3.13}$$

for the inverse of 3, so that $b_{2j} = 0$ and $b_{2j+1} = 1$. Therefore, if $c(n,3)$ is odd, the theorem now shows that $a_j = 0$ for j odd, as claimed. □

More generally, the discussion of $\nu_p(c(n,3)) = 0$ is divided according to the residue of p modulo 3. This division is a consequence of the fact that for $p = 3u+1$, we have

$$i_{p^\infty}(3) = 1 + 2u \sum^{\infty} p^m, \tag{3.14}$$

and for $p = 3u+2$, one computes $p^2 = 3(3u^2+4u+1)+1$, to conclude that

$$i_{p^\infty}(3) = 1+2(3u^2+4u+1)\sum_{m=0}^{\infty} p^{2m} = 1+\sum_{m=0}^{\infty} up^{2m}+(2u+1)p^{2m+1}. \tag{3.15}$$

Theorem 3.5. *Let $p \neq 3$ be a prime and $n = a_0 + a_1p + a_2p^2 + \ldots + a_dp^d$ as before. Then p does not divide $c(n,3)$ if and only if the p-adic digits of n satisfy*

$$a_j < \begin{cases} p/3 & \text{if } j \text{ is odd or } p = 3u+1, \\ 2p/3 & \text{otherwise.} \end{cases} \tag{3.16}$$

For general k we have the following analogous statement.

Theorem 3.6. *Let $p = ku+1$ be a prime. Then p does not divide $c(n,k)$ if and only if the p-adic digits of n are less than p/k.*

Observe that Theorem 3.6 implies the following well-known property of the central binomial coefficients: C_n is not divisible by $p \neq 2$ if and only if the p-adic digits of n are less than $p/2$.

Now we return to (2.9) which will be written as

$$\nu_p(c(n,k)) = \sum_{\alpha \geq 0} \left\lfloor \tfrac{1}{p^{\alpha+1}} \textstyle\sum_{m=0}^{\alpha} (a_m+b_m)p^m \right\rfloor. \tag{3.17}$$

From here, we bound

$$\sum_{m=0}^{\alpha}(a_m+b_m)p^m \leq \sum_{m=0}^{\alpha}(2p-2)p^m = 2(p^{\alpha+1}-1) < 2p^{\alpha+1}. \tag{3.18}$$

Therefore, each summand in (3.17) is either 0 or 1. The p-adic valuation of $c(n,p)$ counts the number of 1's in this sum. This proves the final result.

Theorem 3.7. *Let p be a prime that does not divide k. Then, with the previous notation for a_m and b_m, we have that $\nu_p(c(n,k))$ is the number of indices m such that either*

- *$a_m + b_m \geq p$ or*
- *there is $j \leq m$ such that $a_{m-i} + b_{m-i} = p-1$ for $0 \leq i \leq j-1$ and $a_{m-j} + b_{m-j} \geq p$.*

Corollary 3.8. *Let p be a prime that does not divide k, and write $n = \sum a_m p^m$ and $i_{p^\infty}(k) = 1 + \sum b_m p^m$, as before. Let v_1 and v_2 be the number of indices m such that $a_m + b_m \geq p$ and $a_m + b_m \geq p-1$, respectively. Then*

$$v_1 \leq \nu_p(c(n,k)) \leq v_2. \tag{3.19}$$

4. A q-generalization of $c(n,k)$

A standard procedure to generalize an integer expression is to replace $n \in \mathbb{N}$ by the polynomial

$$[q]_n := \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}. \tag{4.1}$$

The original expression is recovered as the limiting case $q \to 1$. For example, the factorial $n!$ is extended to the polynomial

$$[n]_q! := [n]_q[n-1]_q \ldots [2]_q[1]_q = \prod_{j=1}^{n} \frac{1-q^j}{1-q}. \tag{4.2}$$

The reader will find in [5] an introduction to this q-world.

In this spirit we generalize the integers

$$c(n,k) = \frac{k^n}{n!} \prod_{m=0}^{n-1} (km+1) = \prod_{m=1}^{n} \frac{k(k(m-1)+1)}{m}, \tag{4.3}$$

into the q-world as

$$F_{n,k}(q) := \prod_{m=1}^{n} \frac{[km]_q[k(m-1)+1]_q}{[m]_q^2}. \tag{4.4}$$

Note that this expression indeed gives $c(n,k)$ as $q \to 1$. The corresponding extension of Theorem 2.2 is stated in the next result. The proof is similar to that given above, so it is left to the curious reader.

Theorem 4.1. *The function*

$$F_{n,k}(q) := \prod_{m=1}^{n} \frac{(1-q^{km})(1-q^{k(m-1)+1})}{(1-q^m)^2} \tag{4.5}$$

is a polynomial in q with integer coefficients.

5. Future directions

In this final section we discuss some questions related to the integers $c(n,k)$.

• **A combinatorial interpretation**. The integers $c(n,2)$ are given by the central binomial coefficients $C_n = \binom{2n}{n}$. These coefficients appear in many counting situations: C_n gives the number of walks of length $2n$ on an infinite linear lattice that begin and end at the origin. Moreover, they provide the exact answer for the elementary sum

$$\sum_{k}^{n} \binom{n}{k}^2 = C_n. \tag{5.1}$$

Is it possible to produce similar results for $c(n,k)$, with $k \neq 2$? In particular, what do the numbers $c(n,k)$ count?

• **A further generalization**. The polynomial $F_{n,k}(q)$ can be written as

$$F_{n,k}(q) = \frac{(1-q)}{(1-q^{kn+1})} \prod_{m=1}^{n} \frac{(1-q^{km})(1-q^{km+1})}{(1-q^m)^2} \tag{5.2}$$

which suggests the extension

$$G_{n,k}(q,t) := \frac{(1-q)}{(1-tq^{kn})} \prod_{m=1}^{n} \frac{(1-q^{km})(1-tq^{km})}{(1-q^m)^2} \tag{5.3}$$

so that $F_{n,k}(q) = G_{n,k}(q,q)$. Observe that $G_{n,k}(q,t)$ is not always a polynomial. For example,

$$G_{2,1}(q,t) = \frac{1-qt}{1-q^2}. \tag{5.4}$$

On the other hand,

$$G_{1,2}(q,t) = q+1. \tag{5.5}$$

The following functional equation is easy to establish.

Proposition 5.1. *The function $G_{n,k}(q,t)$ satisfies*

$$G_{n,k}(q,tq^k) = \frac{(1-q^{kn}t)}{(1-q^kt)} G_{n,k}(q,t). \tag{5.6}$$

The reader is invited to explore its properties. In particular, find minimal conditions on n and k to guarantee that $G_{n,k}(q,t)$ is a polynomial.

Consider now the function

$$H_{n,k,j}(q) := G_{n,k}(q,q^j) \tag{5.7}$$

that extends $F_{n,k}(q) = H_{n,k,1}(q)$. The following statement predicts the situation where $H_{n,k,j}(q)$ is a polynomial.

Problem. Show that $H_{n,k,j}(q)$ is a polynomial precisely if the indices satisfy $k \equiv 0 \bmod \gcd(n,j)$.

• **A result of Erdös, Graham, Ruzsa and Strauss**. In this paper we have explored the conditions on n that result in $\nu_p(c(n,k)) = 0$. Given two distinct primes p and q, P. Erdös et al. [2] discuss the existence of indices n for which $\nu_p(C_n) = \nu_q(C_n) = 0$. Recall that by Theorem 3.6 such numbers n are characterized by having p-adic digits less than $p/2$ and q-adic digits less than $q/2$. The following result of [2] proves the existence of infinitely many such n.

Theorem 5.2. *Let A, $B \in \mathbb{N}$ such that $A/(p-1)+B/(q-1) \geq 1$. Then there exist infinitely many numbers n with p-adic digits $\leq A$ and q-adic digits $\leq B$.*

This leaves open the question for $k>2$ whether or not there exist infinitely many numbers n such that $c(n,k)$ is neither divisible by p nor by q. The extension to more than two primes is open even in the case $k=2$. In particular, a prize of \$1000 has been offered by R. Graham for just showing that there are infinitely many n such that C_n is coprime to $105 = 3 \cdot 5 \cdot 7$. On the other hand, it is conjectured that there are only finitely many indices n such that C_n is not divisible by any of 3, 5, 7 and 11.

Finally, we remark that Erdös et al. conjectured in [2] that the central binomial coefficients C_n are never squarefree for $n > 4$ which has been proved by Granville and Ramare in [4]. Define

$$\tilde{c}(n,k) := \text{Numerator}\left(k^{-n}c(n,k)\right). \tag{5.8}$$

We have *some* empirical evidence which suggests the existence of an index $n_0(k)$, such that $\tilde{c}(n,k)$ is not squarefree for $n \geq n_0(k)$. The value of $n_0(k)$ could be large. For instance

$$\begin{aligned}\tilde{c}(178,5) \quad = \quad & 10233168474238806048538224953529562250076040177895261 \\ & 58561031939088200683714293748693318575050979745244814 \\ & 765545543340634517536617935393944411414694781142\end{aligned}$$

is squarefree, so that $n_0(5) \geq 178$. The numbers $\tilde{c}(n,k)$ present new challeges, even in the case $k=2$. Recall that $\frac{1}{2}C_n$ is odd if and only if n is a power of 2. Therefore, C_{786} is not squarefree. On the other hand, the complete factorization of C_{786} shows that $\tilde{c}(786,2)$ is squarefree. We conclude that $n_0(2) \geq 786$.

Acknowledgments. The work of the second author was partially funded by NSF-DMS 0409968. The first author was partially supported, as a graduate student, by the same grant.

References

[1] G. Boros and V. Moll. *Irresistible Integrals*. Cambridge University Press, New York, 1st edition, 2004.

[2] P. Erdös, R. Graham, I. Ruzsa, and E. Straus. On the prime factors of $\binom{2n}{n}$. *Math. Comp.*, 29:83–92, 1975.

[3] F. Gouvea. *p-adic numbers*. Springer Verlag, 2nd edition, 1997.

[4] A. Granville and O. Ramare. Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients. *Mathematika*, 43:73–107, 1996.

[5] V. Kac and P. Cheung. *Quantum Calculus*. Springer-Verlag Universitytext, New York, 2002.

[6] W. Lang. On generalizations of the Stirling numbers. *Journal of Integer Sequences*, 3:Article 00.2.4; 18 pages, 2000.
[7] A. M. Legendre. *Theorie des Nombres*. Firmin Didot Freres, Paris, 1830.
[8] M. Ram Murty. *Introduction to p-adic Analytic Number Theory*, volume 27 of *Studies in Advanced Mathematics*. American Mathematical Society, 1st edition, 2002.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: astraub@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: tamdeberhan@math.tulane.edu