

DYNAMICS OF THE DEGREE SIX LANDEN TRANSFORMATION

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ABSTRACT. We establish the basin of attraction for the fixed point $(3, 3)$ of a dynamical system arising from the evaluation of a definite integral.

1. **Introduction.** The transformation theory of elliptic integrals was initiated by Landen in [6, 7], wherein he proved the invariance of the function

$$G(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \quad (1)$$

under the transformation

$$a_1 = (a + b)/2 \quad b_1 = \sqrt{ab} \quad (2)$$

i.e. that

$$G(a_1, b_1) = G(a, b). \quad (3)$$

Gauss [4] rediscovered the invariance of $G(a, b)$ under the transformation (2) while numerically calculating the length of a lemniscate. The Gauss-Landen transformation can be iterated to produce a convergent double sequence (a_n, b_n) that satisfies $0 < a_n - b_n < 2^{-n}$. The common limit is the famous *arithmetic-geometric mean* of a and b denoted by $AGM(a, b)$. Passing to the limit in $G(a, b) = G(a_n, b_n)$ yields

$$\frac{\pi}{2 AGM(a, b)} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}. \quad (4)$$

Thus the elliptic integral $G(a, b)$ can be evaluated as a limit of a recursively defined sequence. Information about these topics can be obtained in [3].

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The existence of a Landen transformation for the integral

$$U_6(a, b; c, d, e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx \quad (5)$$

was established in [1]. Indeed the integral U_6 is invariant under the transformation

$$\begin{aligned} a_{n+1} &= \frac{a_n b_n + 5a_n + 5b_n + 9}{(a_n + b_n + 2)^{4/3}} \\ b_{n+1} &= \frac{a_n + b_n + 6}{(a_n + b_n + 2)^{2/3}} \\ c_{n+1} &= \frac{c_n + d_n + e_n}{(a_n + b_n + 2)^{2/3}} \\ d_{n+1} &= \frac{(b_n + 3)c_n + 2d_n + (a_n + 3)e_n}{a_n + b_n + 2} \\ e_{n+1} &= \frac{c_n + e_n}{(a_n + b_n + 2)^{1/3}}. \end{aligned} \quad (6)$$

The first two equations in (6) are independent of the variables c , d and e so they define a map

$$\Phi_6(a, b) = \left(\frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}}, \frac{a + b + 6}{(a + b + 2)^{2/3}} \right) \quad (7)$$

that is well-defined on \mathbb{R}^2 minus the line $a + b + 2 = 0$. The goal of this paper is to provide a purely dynamical proof of

Main Theorem: *The basin of attraction for the fixed point $(3, 3)$ of the dynamical system*

$$\begin{aligned} a_{n+1} &= \frac{a_n b_n + 5a_n + 5b_n + 9}{(a_n + b_n + 2)^{4/3}} \\ b_{n+1} &= \frac{a_n + b_n + 6}{(a_n + b_n + 2)^{2/3}} \end{aligned} \quad (8)$$

is the region of the (a, b) -plane for which the integral (5) converges.

Section 2 describes the origin of these transformations: they appear from a sequence of elementary changes of variables that preserve a rational integral. In section 3 we prove that the sequence (a_n, b_n) converges to $(3, 3)$ provided the initial point is on the first quadrant. Section 4 contains a description of the region Λ on the (a, b) -plane on which the integral U_6 converges. This is given in terms of the **discriminant curve** \mathfrak{R} defined by

$$R(a, b) := 4a^3 + 4b^3 - 18ab - a^2b^2 + 27 = 0. \quad (9)$$

Section 5 describes a relation between the curve \mathfrak{R} and the range of the map Φ_6 that defines (8). This gives a rational parametrization of \mathfrak{R} that is used to describe the dynamics of Φ_6 on it. Section 6 presents the fixed points of Φ_6 and their linearizations. Finally Section 7 presents the proof of the main result.

Then the set $R(a, b) = 0$ is a real algebraic curve with two connected components \mathfrak{R}_\pm . The component \mathfrak{R}_+ , of equation $R_+(a, b) = 0$ is contained in the first

quadrant and contains the point $(3, 3)$ as a cusp. The second component \mathfrak{R}_- given by $R_-(a, b) = 0$, is disjoint from the first quadrant.

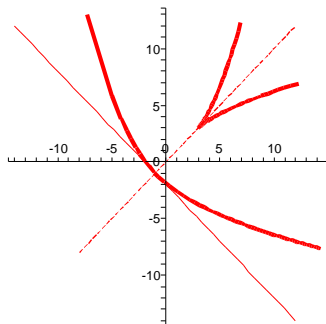


FIGURE 1. The discriminant curve

The identity

$$R(a_1, b_1) = \frac{(a - b)^2 R(a, b)}{(a + b + 2)^4}, \quad (10)$$

plays an important role in the dynamics of the (8). The proof is elementary. In particular it follows from here that the discriminant locus $R(a, b) = 0$, and the regions $\{(a, b) : R(a, b) > 0\}$, located between the two branches in figure 1, and $\{(a, b) : R(a, b) < 0\}$ are preserved by Φ_6 .

The identity (10) also shows that the diagonal $\Delta = \{(a, b) : a = b\}$ of \mathbb{R}^2 is mapped onto the discriminant curve \mathfrak{R} . This yields the parametrization

$$\begin{aligned} a(t) &= \frac{t + 9}{2^{4/3}(t + 1)^{1/3}} \\ b(t) &= \frac{2^{1/3}(t + 3)}{(t + 1)^{2/3}} \end{aligned}$$

of this curve. This parametrization will be obtained in Section 5 as a consequence of the analysis of mapping properties of Φ_6 .

The elementary result of Section 2 states that

Theorem 1. *The boundary of the convergent set Λ is the curve \mathfrak{R}_- . Thus*

$$U_6(a, b; c, d, e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx \quad (11)$$

is finite if and only if $R_-(a, b) > 0$.

The method described in Section 2 was used in [2] to produce a Landen transformation for the integral of any even rational function, that is, given an even rational function R there is a new one R_+ such that

$$\int_0^\infty R(x) dx = \int_0^\infty R_+(x) dx. \quad (12)$$

For example, the integral

$$U_8(a, b, c; d, e, f, g) := \int_0^\infty \frac{dx^6 + ex^4 + fx^2 + g}{x^8 + ax^6 + bx^4 + cx^2 + 1} dx \quad (13)$$

is invariant under the transformation

$$\begin{aligned} a_{n+1} &= \frac{b_n(a_n + c_n) + 4a_n c_n + 10(a_n + c_n) + 8(b_n + 2)}{(a_n + b_n + c_n + 2)^{3/2}} \\ b_{n+1} &= \frac{a_n c_n + 6(a_n + c_n) + 2(b_n + 10)}{a_n + b_n + c_n + 2} \\ c_{n+1} &= \frac{a_n + c_n + 8}{(a_n + b_n + c_n + 2)^{1/2}} \end{aligned} \quad (14)$$

for the parameters in the denominator and

$$\begin{aligned} d_{n+1} &= \frac{d_n + e_n + f_n + g_n}{(a_n + b_n + c_n + 2)^{3/4}} \\ e_{n+1} &= \frac{g_n(3a_n + b_n + 6) + f_n(a_n + 4) + e_n(c_n + 4) + d_n(3c_n + b_n + 6)}{(a_n + b_n + c_n + 2)^{5/4}} \\ f_{n+1} &= \frac{g_n(a_n + 5) + f_n + e_n + d_n(c_n + 5)}{(a_n + b_n + c_n + 2)^{3/4}} \\ g_{n+1} &= \frac{g_n + d_n}{(a_n + b_n + c_n + 2)^{1/4}}, \end{aligned}$$

for the numerator parameters.

The convergence of this procedure has been established in [5] by showing that $R_+(x)dx$ is the direct image of the 1-form $\varphi = R(x)dx$ under the map

$$w = \pi(z) = \frac{z^2 - 1}{2z}. \quad (15)$$

In terms of the sections $\sigma_\pm(w) = w \pm \sqrt{w^2 + 1}$ the new form is

$$\pi_* \varphi = R(\sigma_+(w)) \frac{d\sigma_+}{dw} + R(\sigma_-(w)) \frac{d\sigma_-}{dw}. \quad (16)$$

The main result of [5] is that the dynamical system linked to these transformations converges precisely on the region where the original integral is finite. In this paper we provide a purely dynamical proof of this result, in the case of degree 6.

2. The transformation for U_6 . A polynomial $P_d(x)$ of degree d is called *symmetric* if $P_d(1/x) = x^{-d}P_d(x)$. A symmetric polynomial $P_d(x)$ is said to be *normalized* if it is monic. For example the normalized polynomial of degree 6 is $P_6(x) = x^6 + ax^4 + ax^2 + 1$ and that of degree 12 is

$$P_{12}(x) = (x^{12} + 1) + a_3(x^{10} + x^2) + a_2(x^8 + x^4) + 2a_1x^6.$$

The first step in the derivation of (6) is to symmetrize the denominator of the integrand, producing an integral in which the degree of the denominator is double that of the original. We then employ a sequence of elementary substitutions to transform the new integral back to one with the original degree.

Proposition 1. *Let $R_4(x) = cx^4 + dx^2 + e$, $Q_6(x) = x^6 + ax^4 + bx^2 + 1$, $R_{10}(x) = R_4(x)(x^6 + bx^4 + ax^2 + 1)$ and $P_{12}(x)$ be the normalized polynomial of degree 12. Then*

$$\int_0^\infty \frac{R_4(x)}{Q_6(x)} dx = \int_0^\infty \frac{R_{10}(x)}{P_{12}(x)} dx. \quad (1)$$

Proof. Observe that $P_{12}(x) = x^6 Q_6(x) Q_6(1/x)$ and $R_{10}(x) = x^6 R_4(x) Q_6(1/x)$. \square

Now transform the integral in (1) using the change of variables $x = \tan \theta$ to produce

$$U_6 = \int_0^{\pi/2} \left(\sum_{k=0}^5 r_k \cos^k 2\theta \right) \times \left(\sum_{k=0}^3 s_{2k} \cos^{2k} 2\theta \right)^{-1} 2d\theta,$$

where r_0, \dots, r_5 and s_0, \dots, s_6 are functions of the parameters a, \dots, e . For example, $r_0 = 2c + ac + bc + 2d + ad + bd + 2e + ae + be$, with similar expressions for the rest of them. Observe that the denominator is an even function of $\cos 2\theta$, so the terms with odd powers in the numerator have vanishing integral. Therefore, with $\psi = 2\theta$, we have

$$U_6 = 2 \int_0^{\pi/2} \frac{r_4 \cos^4 \psi + r_2 \cos^2 \psi + r_0}{s_6 \cos^6 \psi + s_4 \cos^4 \psi + s_2 \cos^2 \psi + s_0} d\psi.$$

Letting $\theta = 2\psi$, we obtain

$$U_6 = \int_0^\pi \frac{t_2 \cos^2 \theta + t_1 \cos \theta + t_0}{u_3 \cos^3 \theta + u_2 \cos^2 \theta + u_1 \cos \theta + u_0} d\theta,$$

where t_2, \dots, t_0 and u_3, \dots, u_0 are again functions of the original parameters in U_6 . Finally, the change of variables $y = \tan(\theta/2)$ yields

$$U_6 = \int_0^\infty \frac{v_4 y^4 + v_2 y^2 + v_0}{w_6 y^6 + w_4 y^4 + w_2 y^2 + w_0} dy,$$

with v_4, \dots, v_0 and w_6, \dots, w_0 dependent upon a, b, c, d and e . The last step in the proof of (6) is to factor out w_0 and scale y to produce a monic polynomial in the denominator of the integrand.

3. Convergence for positive initial data. The first two equations of (6):

$$\begin{aligned} a_{n+1} &= \frac{a_n b_n + 5a_n + 5b_n + 9}{(a_n + b_n + 2)^{4/3}} \\ b_{n+1} &= \frac{a_n + b_n + 6}{(a_n + b_n + 2)^{2/3}} \end{aligned} \quad (1)$$

are independent of c, d and e so they may be considered as a dynamical system on \mathbb{R}^2 . In this section we prove that $(a_n, b_n) \rightarrow (3, 3)$ provided $a_0, b_0 \geq 0$. The proof is a slight improvement over the one discussed in [1].

Theorem 2. *Let $a_0 \geq 0$ and $b_0 \geq 0$. Then the sequence (a_n, b_n) defined in (8) converges to $(3, 3)$.*

Proof. It suffices to prove that

$$(a_1 - 3)^2 + (b_1 - 3)^2 \leq \frac{1}{2} [(a_0 - 3)^2 + (b_0 - 3)^2], \quad (2)$$

since iterating this inequality produces

$$[(a_n - 3)^2 + (b_n - 3)^2] \leq 2^{-n} [(a_0 - 3)^2 + (b_0 - 3)^2]$$

and we then have geometric convergence to $(3, 3)$.

The inequality (2) is equivalent to

$$\begin{aligned} f(a, b) = & (a+b+2)^{8/3}(a^2+b^2-6a-6b-18)+2(a+b+2)^{4/3}(4ab+18a+18b+18-a^2-b^2)+ \\ & +2(6a^3+6b^3+8a^2b+8ab^2+35a^2+35b^2-a^2b^2+78a+78b+52ab+63) \geq 0, \end{aligned}$$

and we need to prove that $f(a, b)$ has an absolute minimum of 0 at $(3, 3)$. Note that $f(a, b) = f(b, a)$, so we may restrict the analysis to the region

$$\Omega = \{(a, b) \in \mathbb{R}_+^2 : a \geq b\}. \quad (3)$$

Introduce the new variables $x = (a + b + 2)^{1/3}$ and $y = ab$, and write $h(x, y)$ for $f(a, b)$. The region Ω is then transformed into

$$\Omega^* = \{(x, y) \in \mathbb{R}_+^2 : x \geq \sqrt[3]{2} \text{ and } 0 \leq y \leq (1 - x^3/2)^2\},$$

and in terms of the new variables, we need to prove that

$$\begin{aligned} h(x, y) = & x^{14} - 10x^{11} - 2x^{10} + 12x^9 - 2x^8(y+1) + 44x^7 - \\ & - 2x^6 + 4x^4(3y-11) - 20x^3(y-1) - 2(y-1)^2 \geq 0 \end{aligned}$$

for $(x, y) \in \Omega^*$. This will be achieved by showing that $h \geq 0$ on the upper part of the boundary of Ω^* and that $h_y(x, y) < 0$.

First observe that

$$\begin{aligned} h_y(x, y) &= -2(2y + x^8 - 6x^4 + 10x^3 - 2) \\ &= -4y - 2P(x - 1) \end{aligned}$$

with

$$P(x) = x^8 + 8x^7 + 28x^6 + 56x^5 + 64x^4 + 42x^3 + 22x^2 + 14x + 3.$$

Thus $h_y(x, y) < 0$ for $(x, y) \in \Omega^*$.

Now we examine the values of h on the upper part of the boundary of Ω^* . Along the curve $y = (1 - x^3/2)^2$, $x \geq \sqrt[3]{2}$, we have

$$\begin{aligned} h(x, (1-x^3/2)^2) = & \frac{1}{8}x^4(x-2)^2 \times (4(x-1)^8 + 48(x-1)^7 + 271(x-1)^6 + 902(x-1)^5 + \\ & + 1905(x-1)^4 + 2628(x-1)^3 + 2289(x-1)^2 + 1062(x-1) + 107), \end{aligned}$$

which has an absolute minimum of 0 at $x = 2$. The proof of the Theorem is complete. \square

Note. We have also shown in [1] that the convergence $(a_n, b_n) \rightarrow (3, 3)$ and the invariance of the integral U_6 under the transformation (6), yield the existence of a

number L such that $(c_n, d_n, e_n) \rightarrow (1, 2, 1)L$. Passing to the limit in the invariance relation

$$\int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx = \int_0^\infty \frac{c_n x^4 + d_n x^2 + e_n}{x^6 + a_n x^4 + b_n x^2 + 1} dx$$

yields

$$\int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx = \frac{\pi}{2} \times L,$$

so that L is essentially the original integral. Thus we obtain an iterative scheme (by keeping track of c_n) to evaluate the integral of a rational function. The method converges quadratically.

4. The region of convergence for the integral U_6 . We now describe the region

$$\Lambda := \left\{ (a, b) \in \mathbb{R}^2 : U_6 := \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx < \infty \right\}, \quad (1)$$

where U_6 converges. Naturally this convergence depends only upon the roots of the denominator, so Λ is independent of the parameters in the numerator. We now provide an elementary characterization of the region Λ .

Step 1: The integral U_6 converges if and only if the polynomial $P(t) := t^3 + at^2 + bt + 1$ has no positive real roots. This follows from the partial fraction decomposition of the integrand.

Step 2: Suppose $a^2 \leq 3b$. Then $(a, b) \in \Lambda$. Observe that if $P(0) = 1$, the condition $a^2 \leq 3b$ implies that P is increasing so it does not have roots in \mathbb{R}^+ .

From now on we assume $a^2 > 3b$.

Step 3: Suppose $a, b > 0$. Then $(a, b) \in \Lambda$. Let $t_+ = \frac{1}{3}(-a + \sqrt{a^2 - 3b})$ be the largest critical point of P . The condition $t_+ < 0$ is equivalent to $a, b > 0$. The conclusion follows from the fact that P is increasing for $t > t_+$ and $P(0) = 1$.

From now on we assume $t_+ > 0$.

Step 4: Suppose $a^2 > 3b$ and $t_+ > 0$. Then the condition $(a, b) \in \Lambda$ is equivalent to $P(t_+) > 0$. By Step 3 we only need to consider the case $t_+ > 0$. The equivalence of the two conditions is clear from the graph of P for $t > 0$. The condition $P(t_+) > 0$ reads

$$27 + 2a^3 - 9ab > 2(a^2 - 3b)^{3/2}. \quad (2)$$

Therefore, if $27 + 2a^3 - 9ab < 0$ then $(a, b) \notin \Lambda$. In the other case, squaring (2) yields $R_-(a, b) > 0$.

The previous steps give the convergence condition stated in Theorem 1.

5. The range of Φ_6 . The study of mapping properties of Φ_6 is facilitated by studying the image of the line

$$L_c := \{(a, b) \in \mathbb{R}^2 : a + b = c\}. \quad (1)$$

It is evident that L_c is mapped to a horizontal segment and the discriminant curve reappears as the curve formed by the endpoints of these segments. This description of \mathfrak{R} requires a classical criterion for the roots of a cubic polynomial:

Cubic root criterion: the nature of the roots of $x^3 + ax + b = 0$ is determined by the discriminant $D = b^2/4 + a^3/27$. In detail

$$\text{Nature of roots} = \begin{cases} D > 0 & \rightarrow \text{one real root, two complex conjugate} \\ D = 0 & \rightarrow \text{three real roots at least two equal} \\ D < 0 & \rightarrow \text{three real and distinct roots.} \end{cases}$$

Theorem 3. *Let $c \in \mathbb{R}$ and define*

$$a_c := \frac{c + 18}{4(c + 2)^{1/3}} \quad \text{and} \quad b_c := \frac{c + 4}{(c + 2)^{2/3}}. \quad (2)$$

The line L_c is mapped to the part of the horizontal line $b = b_c$ above the segment $(-\infty, a_c)$. The point (a_c, b_c) is on the discriminant curve \mathfrak{R} and it provides a parametrization for it. For $-\infty < c \leq -2$ the image ends on the left branch \mathfrak{R}_- ; for $-2 \leq c \leq 6$ the image ends on the left branch of the cusp piece \mathfrak{R}_+ and for $6 \leq c < +\infty$ it ends on the right cusp piece.

Proof. From the expression for Φ_6 it is clear that $\Phi_6(L_c)$ is contained in the horizontal line

$$b \equiv b_c := \frac{c + 4}{(c + 2)^{2/3}}, \quad (3)$$

and the part of this line that is actually achieved is parametrized by

$$a_1(c) = -\frac{a^2 - ac - (5c + 9)}{(c + 2)^{4/3}}.$$

We conclude that Φ_6 folds the line L_c over the horizontal segment $(-\infty, a_c)$ at height b_c .

The function b_c maps $\mathbb{R} - \{-2\}$ onto \mathbb{R} and given a value of b , the parameter c is determined by the cubic equation

$$t^3 + (12 - b^3)t^2 + 48t + 64 = 0, \quad (4)$$

that normalizes to

$$t^3 + (8b^3 - b^6/3)t + (-16b^3 + 8b^6/3 - 2b^9/27) = 0 \quad (5)$$

of discriminant

$$D = -\frac{64}{27}(b - 3)b^6(b^2 + 3b + 9). \quad (6)$$

By the cubic root criterion 4) has exactly one real root for $b < 3$ and three distinct roots for $b > 3$. The proof is complete. \square

Note. The points

$$a_c = \frac{c + 18}{4(c + 2)^{1/3}} \quad \text{and} \quad b_c = \frac{c + 6}{(c + 2)^{2/3}} \quad (7)$$

are on the discriminant curve $R(a, b) = 0$. They provide a parametrization of this curve. A simpler version of this parametrization is given by

$$a(s) = \frac{s^3 + 4}{s^2} \quad \text{and} \quad b(s) = \frac{s^3 + 16}{4s}. \quad (8)$$

The dynamics of (8) on the discriminant curve can be given explicitly in terms of (8). The next result can be verified directly.

Theorem 4. *The function*

$$\varphi(s) = \left(\frac{4(s^2 + 4)^2}{s(s + 2)^2} \right)^{1/3} \quad (9)$$

satisfies

$$\Phi_6(a(s), b(s)) = (a(\varphi(s)), b(\varphi(s))). \quad (10)$$

6. Fixed points. The goal of this section is to analyze the fixed points of the map Φ_6 defined in (7). These points satisfy

$$\frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}} = a \quad (1)$$

$$\frac{a + b + 6}{(a + b + 2)^{2/3}} = b. \quad (2)$$

Lemma 1. *There are three fixed points of Φ_6 . They are given by*

1. $P_1 := (3, 3)$,
2. $P_2 := (-3 + r - 2r^2, r^2 - 1) \approx (-4.205569, 3.957741)$ where $r \approx 1.353210$ is a root of $z^3 + z^2 - z - 2$,
3. $P_3 := (-3 + r, 3 + 2r - r^2) \approx (-5.309144, 0.8311772)$ where $r \approx 1.205569$ is a root of $z^3 - z^2 - z - 2$.

Moreover, $R(P_1) = 0$, $R(P_2) = 0$, and $R(P_3) < 0$. The point P_1 is super-attracting, P_2 is a saddle point, and P_3 is a stable spiral.

Proof. Introduce the auxiliary variable $z = (a + b + 2)^{1/3}$. Then (2) yields

$$\begin{aligned} a &= z^3 - 2 - z^{-2}(z^3 + 4) \\ b &= z^{-2}(z^3 + 4), \end{aligned}$$

and then (2) produces

$$\begin{aligned} P(z) &= z^{11} - z^9 - 3z^8 - 5z^7 - 3z^6 - 2z^5 + z^4 + 8z^3 + 8z^2 + 16 \\ &= (z - 2)(z^2 + z - 1)(z^2 + z + 2)(z^3 + z^2 - z - 2)(z^3 + z^2 + z + 2). \end{aligned}$$

The real roots of $P(z) = 0$ are therefore $z_1 = 2$, yielding P_1 , and two roots coming from the cubic factors that produce the other two fixed points. Direct calculation shows that two of the fixed points are on the resolvent curve.

The linearization Φ_6 at (a, b) is given by the matrix

$$J(a, b) = \begin{pmatrix} \frac{3b^2 - ab + b - 5a - 6}{3(a+b+2)^{7/3}} & \frac{3a^2 - ab + a - 5b - 6}{3(a+b+2)^{7/3}} \\ \frac{a + b - 6}{3(a+b+2)^{5/3}} & \frac{a + b - 6}{3(a+b+2)^{5/3}} \end{pmatrix} \quad (3)$$

of determinant

$$\text{Det}(J(a, b)) = \frac{(b - a)(a + b - 6)}{3(a + b + 2)^3}. \quad (4)$$

This shows that P_1 is asymptotically stable, P_2 is a saddle point and P_3 is a spiral. This follows by a direct computation of the eigenvalues of $J(a, b)$ at each of these points. \square

Lemma 2. *The only 2-cycles above the line $a + b + 2 = 0$ are fixed points.*

Proof. Suppose $\Phi_6(a, b) = (c, d)$ and $\Phi_6(c, d) = (a, b)$. We need to show that $a = c$ and $b = d$. Letting $m = (a + b + 2)^{1/3}$ and $n = (c + d + 2)^{1/3}$ yields

$$\begin{aligned} a &= m^3 - 2 - \frac{n^3 + 4}{n^2} \\ b &= \frac{n^3 + 4}{n^2} \\ c &= n^3 - 2 - \frac{m^3 + 4}{m^2} \\ d &= \frac{m^3 + 4}{m^2} \end{aligned}$$

Substituting these into the first components of Φ_6 , namely

$$\begin{aligned} a &= \frac{cd + 5c + 5d + 9}{n^4} \\ c &= \frac{ab + 5a + 5b + 9}{m^4}, \end{aligned}$$

yields two equations whose difference has the factor $n - m$ and the term

$$\begin{aligned} &n^4 m(m^5 - m^2 + 1) + m^4 n(n^5 - n^2 + 1) + mn((mn)^3 - 5(mn)^2 + 6mn + 1) \\ &\quad + mn((m^2 n^2 - 2)^2 + m^2(2n^2 - 3n + 2) + n^2(2m^2 - 3m + 2) + 3) \\ &\quad + 8m + 8n + 2m^4 + m^3 + 8m^2 + n^3 + 8n^2 + 2n^4 + m^5 + n^5 + n^2 m + nm^2. \end{aligned}$$

The brackets indicate why this expression is positive when $m, n > 0$. This forces $m = n$, hence $a = c$ and $b = d$. \square

7. The region of convergence of the iteration. In Section 3 we have shown that the map (1) converges to $(3, 3)$ if the initial data (a_0, b_0) is in the first quadrant. We now describe the full region of convergence of this dynamical system.

It is easier to visualize this basin of attraction by sketching the set $\mathfrak{R} := \{(x, y) : R(x, y) = 0\}$. We divide this curve into five pieces, $\{Z_k\}_{k=1}^5$, omitting the endpoints $(3, 3)$, P_2 and $(-1, -1)$; see figure 2.

The main theorem claims that the basin of attraction of $(3, 3)$ is the set of points above the part of the curve $R = 0$ which contains Z_1, Z_2, Z_3 , that is, the curve $\mathfrak{R}_- := Z_1 \cup P_2 \cup Z_2 \cup (-1, -1) \cup Z_3$. This is the region Λ in Section 2.

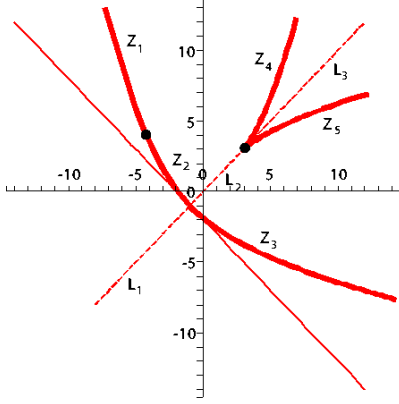
The proof will require a technical lemma, that we prove next.

Lemma 3. *The function*

$$f(x, y, w) := \frac{(1+w)(x+y-6)(x+y+2)^{2/3}}{3(x+y+2)(xw+y+5+5w) - 4(xy+5x+5y+9)(1+w)}$$

satisfies $-1 \leq f(x, y, w) \leq 0$ on the set D , where

$$D := \{(x, y, w) : -4.206 \leq x \leq 0, 3 \leq y \leq 3.96, -1 \leq w \leq 0\}$$

FIGURE 2. The fixed points of Φ_6

Proof. The numerator of f is always negative in D . Let

$$g(x, y, w) := 3(x + y + 2)(xw + y + 5 + 5w) - 4(xy + 5x + 5y + 9)(1 + w)$$

be the denominator of f . We first show that $g > 0$ on D . It suffices to consider g when $w = 0$ and $w = -1$ since g is linear in w . On D , we have

$$g(x, y, 0) = -6 - xy - 5x + 3y^2 + y > 0$$

and

$$g(x, y, -1) = -x^2 - 2x + y^2 + 2y = -(x + 1)^2 + (y + 1)^2 > 0.$$

Therefore, g is positive on D . We conclude that $f < 0$ on D .

The inequality $f \geq -1$ on D is equivalent to $h > 0$ on D , where

$$h(x, y, w) := (1 + w)(x + y - 6)(x + y + 2)^{2/3} + 3(x + y + 2)(xw + y + 5 + 5w) - 4(xy + 5x + 5y + 9)(1 + w).$$

Again, since h is linear in w , it suffices to prove this when $w = 0$ and $w = -1$. We have

$$h(x, y, -1) = 3(x + y + 2)(y - x) > 0$$

on D . Lastly,

$$h(x, y, 0) = (x + y - 6)(x + y + 2)^{2/3} - 6 - xy - 5x + 3y^2 + y.$$

The first term is minimized when $x + y = 6/5$, so one easily obtains $h(x, y, 0) > 0$. \square

Define E as the region

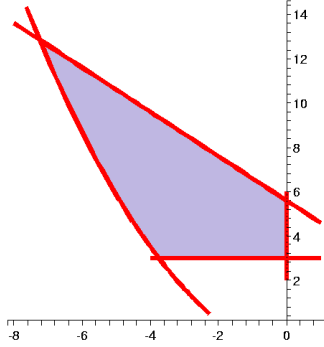
$$E := \{(a, b) : a \leq 0, b \geq 3, a + b \leq 5.6, R(a, b) > 0\},$$

depicted in figure 3.

Lemma 4. *The function Φ_6 is injective on the region E .*

Proof. Suppose $(a, b) \in E$, $(c, d) \in E$ and

$$\Phi_6(a, b) = \Phi_6(c, d). \quad (1)$$

FIGURE 3. The region E .

Letting $u = (a + b + 2)^{1/3}$ and $v = (c + d + 2)^{1/3}$, the geometry of E implies $u, v \in (1, 2)$. The second component of (1) yields

$$u + \frac{4}{u^2} = v + \frac{4}{v^2}$$

so that $(u - v)(u^2v^2 - 4u - 4v) = 0$. If $u \neq v$ we have

$$\begin{aligned} u^2v^2 &= 4u + 4v \\ &\geq 8\sqrt{uv} \end{aligned}$$

hence $uv \geq 4$, contradicting that $u, v < 2$. Thus $u = v$ and (1) now yields

$$a + b = c + d, \quad ab = cd.$$

This forces $ab = c(a + b - c)$, so that $c = a$ or $c = b$. The domain E prohibits $c = b$, hence $c = a$, and subsequently $d = b$. This proves the desired result. \square

Note that while Equation 4 guarantees that Φ_6 is *locally* injective at each point in E , this is not in itself sufficient to give the *global* injectivity of Φ_6 over all of E . For an example of a two-dimensional polynomial which is locally invertible everywhere but not globally invertible, see [8].

Now we give the proof of the main theorem.

Proof. The identity (10) indicates that the set \mathfrak{R}_- is positively invariant and also tied to the line $L := \{(t, t) : t \neq -1\}$. Let L be divided into three pieces, $\{L_k\}_{k=1}^3$, omitting the endpoints $(-1, -1)$ and P_1 . By choosing representatives from each of these eight curves, one has the following observations about where these curves map:

1. L_1 maps onto $Z_1 \cup P_2 \cup Z_2 \cup (-1, -1) \cup Z_3$.
2. L_2 maps onto Z_4 .
3. L_3 maps onto Z_4 .
4. Z_1 maps into $Z_1 \cup P_2 \cup Z_2$.
5. Z_2 maps onto Z_1 .
6. Z_3 maps into $Z_1 \cup P_2 \cup Z_2$.
7. Z_4 maps onto itself.
8. Z_5 maps onto Z_4 .

These relationships imply that the region Λ above \mathfrak{R}_- is invariant and so is the region below this curve. The eigenvalues of the Jacobian of Φ_6 at P_1 are both zero, so this fixed point is not only asymptotically stable but the convergence is asymptotically quadratic. It remains to show that Λ is the basin of attraction for P_1 .

The line $a + b + 2 = 0$ lies below the curve \mathfrak{R}_- except where it is tangent at $(-1, -1)$. This is readily seen since

$$R(a, -a - 2) = -(a + 1)^2(a^2 + 2a + 5).$$

Therefore $a + b + 2 > 0$ for all $(a, b) \in \Lambda$.

Note that the second coordinate of $\Phi_6(a, b)$ equals

$$\frac{a + b + 6}{(a + b + 2)^{2/3}} = w + \frac{4}{w^2} := f(w)$$

where $w = (a + b + 2)^{1/3}$. When w is positive – which we know is true in Λ – the function f obtains a minimum at $w = 2$ with $f(2) = 3$. This means that Λ maps into the region $\tilde{\Lambda} := \Lambda \cap \{(a, b) : b \geq 3\}$, so it suffices to consider only this smaller region.

This argument may be sharpened further. Let $w_n = (a_n + b_n + 2)^{1/3}$, where a_n and b_n are as in the statement of the theorem. Considering only $a_n \leq 0$ and $b_n \geq 3$ gives

$$\begin{aligned} w_{n+1}^3 &= a_{n+1} + b_{n+1} + 2 \\ &\leq \frac{5w_n^3 - 1}{w_n^4} + \frac{w_n^3 + 4}{w_n^2} + 2 \end{aligned}$$

It is straightforward to show that if $w_n > 1.96$, then $w_{n+1} < w_n$, therefore (a_n, b_n) enters the first quadrant, or $w_n \leq 1.96$ eventually. This forces $a_n + b_n \leq 5.6$. We have reduced the problem to showing that all points in E eventually enter the first quadrant.

Since P_2 is a saddle point and the regions above and below the curve \mathfrak{R}_- are invariant, this implies the stable manifold lies in $Z_1 \cup P_2 \cup Z_2$. Indeed, Lemma 2 implies this entire set is the stable manifold.

The positive eigenvalue at P_2 (~ 7.07) has a corresponding eigenvector whose slope is approximately -0.10367 . In a neighborhood of P_2 , the unstable manifold takes the form $(a, \phi(a))$ and its invariance implies the functional equation

$$\phi \left(\frac{a\phi(a) + 5a + 5\phi(a) + 9}{(a + \phi(a) + 2)^{4/3}} \right) = \frac{a + \phi(a) + 6}{(a + \phi(a) + 2)^{2/3}}$$

holds with $\phi'(-4.20557) \approx -0.10367$. Differentiating this equation gives

$$\begin{aligned} \phi' \left(\frac{a\phi(a) + 5a + 5\phi(a) + 9}{(a + \phi(a) + 2)^{4/3}} \right) &= \\ \frac{(1 + w)(a + b - 6)(a + b + 2)^{2/3}}{3(a + b + 2)(aw + b + 5 + 5w) - 4(ab + 5a + 5b + 9)(1 + w)} \end{aligned}$$

and this is the function $f(a, b, w)$ of Lemma 3, where $b = \phi(a)$ and $w = \phi'(a)$. Lemmas 1 and 3 imply that for $-4.206 < a < 0$ the unstable manifold $b = \phi(a)$ is a decreasing function which extends at least to $a = 0$. Therefore, the unstable manifold enters the positively invariant first quadrant.

In Section 3 we have shown that if (a, b) is in the first quadrant, Φ_6 cuts the distance from (a, b) to P_1 by at least half, so the distance function acts as a Liapunov

function for P_1 in this quadrant. We conclude that the unstable manifold from P_2 approaches P_1 .

Lastly, consider the how the set E evolves. Since the portion on the boundary which lies on $R = 0$ approaches P_2 , the parts of the boundary with $b = 3$ and $x + y = 5.6$ maps uniformly close to the unstable manifold. Lemma 4 guarantees that the images of E stay between the images of the lower and upper boundaries, hence all points in E tend to P_1 . This completes the proof.

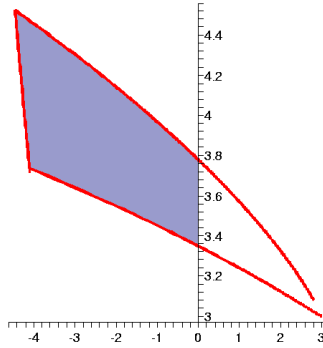


FIGURE 4. Three iterations of the region E .

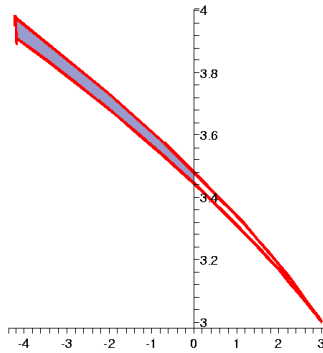


FIGURE 5. Six iterations of the region E .

□

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REFERENCES

- [1] G. Boros and V. Moll, *A rational Landen transformation. The case of degree six*. Contemporary Mathematics, **251** (2000), 83-91.
- [2] G. Boros and V. Moll, *Landen transformation and the integration of rational function*. Math. Comp., **71** (2001), 649-668.
- [3] J. M. Borwein and P. Borwein, *Pi and the AGM*. John Wiley and Sons, 1987.
- [4] K. F. Gauss, *Arithmetische Geometrisches Mittel*, (1799). In *Werke*, **3**, 361-432. Konigliche Gesellschaft der Wissenschaft, Gottingen. Reprinted by Olms, Hildesheim, 1981.
- [5] J. Hubbard and V. Moll, *A geometric view of the rational Landen transformation*. Bull. London Math. Soc., **35** (2003), 293-301.
- [6] J. Landen, *A disquisition concerning certain fluents, which are assignable by the arcs of the conic sections; wherein are investigated some new and useful theorems for computing such fluents*. Philos. Trans. Royal Soc. London, **61** (1771), 298-309.
- [7] J. Landen, *An investigation of a general theorem for finding the length of any arc of any conic hyperbola, by means of two elliptic arcs, with some other new and useful theorems deduced therefrom*. Philos. Trans. Royal Soc. London, textbf65 (1775), 283-289.
- [8] S. I. Pinchuk, *Counterexample to the Strong Real Jacobian Conjecture*. Math. Z., **217** (1994), 1-4.

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