# RATIONAL LANDEN TRANSFORMATIONS ON $\mathbb{R}$

DANTE MANNA AND VICTOR H. MOLL

ABSTRACT. The Landen transformation $(a, b) \mapsto ((a+b)/2, \sqrt{ab})$ preserves the value of an elliptic integral and its iteration produces the classical arithmetic-geometric mean $\mathrm{AGM}(a, b)$. We present analogous transformations for rational functions integrated over the whole real line.

## 1. INTRODUCTION

The problem of indefinite integration of rational functions $R(x) = B(x)/A(x)$ was finished by J. Bernoulli in the eighteenth century. He completed the original attempt by Leibniz of a general partial decomposition of $R(x)$. The result is that a primitive of a rational function is always elementary: it consists of a new rational function (its *rational part*) and the logarithm of a second rational function (its *transcendental part*).

In the middle of the nineteenth century Hermite [13] and Ostrogradsky [19] developed algorithms to compute the rational part of the primitive of $R(x)$ *without* factoring $A(x)$. More recently Horowitz [14] rediscovered this method and discussed its complexity. The problem of computing the transcendental part of the primitive was finally solved by Lazard and Rioboo [16], Rothstein [21] and Trager [22]. For detailed descriptions and proofs of these algorithms the reader is referred to [7] and [11].

This paper contains a method of computing *definite* rational integrals that, unlike the methods described above, does not involve the factorization of any polynomial. In this new method, the value of the integral is obtained as the limit of a sequence of transformations of the coefficients of the integrand. Thus, the algorithm presented here is in the spirit of the classical *Landen transformation* for elliptic integrals. These are integrals of the form

$$(1.1) \qquad K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

that have been studied since the eighteenth century. The reader will find in [18] more information about them. Its trigonometric version,

$$G(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}},$$

was considered by Gauss [9] in his work on the lemniscate. The special case $k = i$,

$$(1.2) \qquad \mathrm{Lem} = \int_0^1 \frac{dx}{\sqrt{1 - x^4}},$$

appears as an expression for its arc length. He inferred from a numerical evaluation that the function $G(a, b)$ is invariant under

$$(1.3) \qquad \mathfrak{E} : (a, b) \quad \mapsto \left( \tfrac{1}{2}(a + b), \sqrt{ab} \right).$$

A transformation of the parameters of an integral is called a *Landen transformation* if it preserves the value of the integral. The example (1.3) is the original one.

It is a classical result that the iteration of $\mathfrak{E}$ produces two sequences, $a_n$ and $b_n$, that converge quadratically to a common limit: $\text{AGM}(a, b)$, the *arithmetic-geometric mean* of $a$ and $b$. The invariance of the elliptic integral (1.2) yields

$$(1.4) \qquad G(a, b) \quad = \quad \frac{\pi}{2 \, \text{AGM}(a, b)}.$$

Iteration of (1.3) provides a method to evaluate the elliptic integral $G(a, b)$. For instance, four steps starting at $a_0 = 1$, $b_0 = \sqrt{2}$ yield 22 correct digits of the integral in (1.2). See [6] for details and its relation to modern evaluations of $\pi$.

We consider here the space of rational functions

$$\mathfrak{R}_p \quad := \quad \left\{ R(x) = \frac{B(x)}{A(x)} \, \middle| \, A(x) = \sum_{k=0}^{p} a_k x^{p-k} \text{ and } B(x) = \sum_{k=0}^{p-2} b_k x^{p-2-k} \right\}.$$

We assume

- The degree $p$ is an even positive integer.
- The coeffients $a_k$ and $b_k$ are real numbers.
- The polynomial $A(x)$ has no real zeros.

Under these assumptions the integral

$$(1.5) \qquad I \quad := \quad \int_{-\infty}^{\infty} R(x) \, dx$$

is finite.

We describe a transformation on the parameters

$$(1.6) \qquad \mathfrak{P}_p \quad := \quad \{ a_0, a_1, \cdots, a_p; b_0, b_1, \cdots, b_{p-2} \}$$

of $R \in \mathfrak{R}_p$ that preserves the integral $I$. In fact, we produce a family of maps, indexed by $m \in \mathbb{N}$,

$$\mathfrak{L}_{m,p} : \mathfrak{R}_p \to \mathfrak{R}_p,$$

such that

$$(1.7) \qquad \int_{-\infty}^{\infty} R(x) \, dx \quad = \quad \int_{-\infty}^{\infty} \mathfrak{L}_{m,p}(R(x)) \, dx.$$

The maps $\mathfrak{L}_{m,p}$ induce a *rational Landen transformation* on the coefficients:

$$(1.8) \qquad \Phi_{m,p} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}.$$

We provide numerical evidence that the iterates of this map converge to a limit, with convergence of order $m$.

In the case $m = p = 2$, we will show that the integral

$$(1.9) \qquad I(a_0, a_1, a_2) \quad = \quad \int_{-\infty}^{\infty} \frac{dx}{a_0 x^2 + a_1 x + a_2}$$

is invariant under the transformation

$$(1.10) \qquad a_0 \;\mapsto\; \frac{2a_0 a_2}{a_0 + a_2},$$

$$a_1 \;\mapsto\; \frac{a_1(a_2 - a_0)}{a_0 + a_2},$$

$$a_2 \;\mapsto\; \frac{(a_0 + a_2)^2 - a_1^2}{2(a_0 + a_2)}.$$

This example is discussed in detail in [17].

The theory of Landen transformations for rational integrands is divided into two cases, according to the domain of integration.

**Case 1**: The interval of integration is not the whole real line.

Integration over a finite interval $[a, b]$ is transformed to the half-line $[0, \infty)$ by a bilinear transformation. In detail,

$$(1.11) \qquad \int_a^b R(x)\, dx \;=\; (b - a) \int_0^\infty R\left(\frac{a + bt}{1 + t}\right) \frac{dt}{(1 + t)^2}.$$

Similarly, integration over half-lines $[a, \infty)$ and $(-\infty, a]$ can be reduced to $[0, \infty)$ by translations and reflections. Thus, the interval $[0, \infty)$ encompasses all integrals that fall in this case.

Landen transformations for *even* rational functions on $[0, \infty)$ were established in [5]. For example, the integral

$$(1.12) \qquad U_6(a, b; c, d, e) \;:=\; \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx$$

is invariant under

$$(1.13) \qquad a \;\mapsto\; \frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}}$$

$$b \;\mapsto\; \frac{a + b + 6}{(a + b + 2)^{2/3}},$$

with similar rules for the coefficients $c$, $d$ and $e$.

The map (1.13) can be iterated to produce a sequence $(a_n, b_n; c_n, d_n, e_n)$ with the property

$$(1.14) \qquad U_6(a_n, b_n; c_n, d_n, e_n) \;=\; U_6(a, b; c, d, e).$$

Its convergence was discussed in [4], assuming that the initial conditions $a_0$, $b_0$ are nonnegative. The main result is the existence of a number $L$, depending on the initial data $a_0, \cdots, e_0$, such that $a_n \to 3$, $b_n \to 3$, and $c_n \to L$, $d_n \to 2L$, and $e_n \to L$. The convergence is quadratic.

The positivity condition on initial data was eliminated in [15], where we reinterpret the Landen transformation (1.13) in geometric terms. The new integrand is the direct image of the original one under the map $w = (z^2 - 1)/2z$. In concrete terms, if $R$ is the original integrand and

$$(1.15) \qquad z_\pm(w) \;=\; w \pm \sqrt{w^2 + 1}$$

are the two branches of the inverse of $w$, then the new integrand is given by

$$(1.16) \qquad R(z_+(w))\frac{dz_+}{dw} + R(z_-(w))\frac{dz_-}{dw}.$$

This geometric interpretation extends to the algorithm presented in [5], where an analogue of (1.13) is given for an arbitrary even function. These transformations on the coefficients define a map,

$$(1.17) \qquad \Phi_{2n} : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1},$$

which is the rational analogue of $\mathfrak{E}$ in (1.3). These are named *even rational Landen*. Using this approach, we have established a proof that the iterations of $\Phi_{2n}$ converge precisely when the initial integral is finite.

A purely dynamical proof of convergence of the iterations of $\Phi_{2n}$ is presented in [8], but only for the case of degree 6. The relation between (1.13) and the invariance of the rational integrals is still part of the argument. It is established that the iterations are eventually mapped to the first quadrant, and then the results of [4] are applied. It would be desirable to obtain a proof of convergence completely independent of the integrals that gave origin to these maps.

The existence of this type of transformation for an odd rational integrand is an open question.

**Case 2**: the domain of integration is the real line.

This is the case we present here. We give a Landen transformation for integrals over $\mathbb{R}$. The convergence of the iterations of these maps can be established along the lines of [15], but a more direct analysis is still an open question. The issue of convergence is not discussed here, except for the numerical examples in Section 7.

The new integrands, $\mathfrak{L}_{m,p}(R(x))$, depend on the parameter $p$, the degree of the denominator of the original integrand, and the parameter $m$, the order of convergence. *Both parameters are arbitrary.*

Section 2 presents a preliminary example that illustrates the methods developed in the rest of the paper. Section 3 introduces two families of polynomials that are the basis of the rational Landen transformations. Section 4 consists of some simple trigonometrical identities. The integrand is scaled in Section 5, using the polynomials studied in Section 3. The algorithm leading to the rational Landen transformation is a consequence of the vanishing of a class of integrals. This is presented in Section 6. Examples are given in the last section.

## 2. AN EXAMPLE

We begin with an example of a Landen transformation that introduces the methods described in later sections.

The integral of the rational function

$$(2.1) \qquad R(x) \quad = \quad \frac{x^2 + x + 1}{x^4 + 6x^3 + 29x^2 + 60x + 100}$$

is evaluated as

$$(2.2) \qquad I := \int_{-\infty}^{\infty} R(x)\,dx \quad = \quad \frac{38\,\pi}{31\sqrt{31}},$$

using the factorization

$$(2.3) \qquad x^4 + 6x^3 + 29x^2 + 60x + 100 = (x^2 + 3x + 10)^2.$$

We will produce a new rational function,

$$(2.4) \qquad \mathfrak{L}_{2,4}(R(x)) \quad = \quad \frac{202x^2 + 45x + 97}{400x^4 + 1080x^3 + 2969x^2 + 3024x + 3136},$$

and show that it satisfies

$$(2.5) \qquad \int_{-\infty}^{\infty} \mathfrak{L}_{2,4}(R(x))\, dx \quad = \quad \int_{-\infty}^{\infty} R(x)\, dx.$$

(The notation $\mathfrak{L}_{2,4}$ indicates the degrees of the transformation used to produce this new function. Details are given in Section 6).

The first step is to multiply the denominator,

$$(2.6) \qquad A(x) \quad = \quad x^4 + 6x^3 + 29x^2 + 60x + 100,$$

by

$$(2.7) \qquad Z(x) \quad = \quad 1600x^4 - 960x^3 + 464x^2 - 96x + 16,$$

so that $E(x) = A(x)Z(x)$ can be written as a homogeneous polynomial in the variables

$$(2.8) \qquad P_2(x) = x^2 - 1 \quad \text{and} \quad Q_2(x) = 2x.$$

(These polynomials will be described in Section 3.) In detail,

$$(2.9) \qquad E(x) \quad = \quad \sum_{l=0}^{4} e_l P_2^{4-l}(x) Q_2^l(x),$$

with $e_0 = 1600$, $e_1 = 4320$, $e_2 = 11876$, $e_3 = 12096$, and $e_4 = 12544$. Then, with $C(x) = B(x)Z(x)$, we obtain

$$(2.10) \qquad I \quad = \quad \int_{-\infty}^{\infty} \frac{B(x)}{A(x)}\, dx = \int_{-\infty}^{\infty} \frac{C(x)}{E(x)}\, dx.$$

Now write

$$(2.11) \qquad E(x) \quad = \quad Q_2^4(x) \left( \sum_{l=0}^{4} e_l R_2(x)^{4-l} \right),$$

where

$$(2.12) \qquad R_2(x) \quad = \quad \frac{P_2(x)}{Q_2(x)} = \frac{x^2 - 1}{2x}.$$

We would like to make the change of variables $y = R_2(x)$ in (2.10). The function $R_2(x)$ has a multivalued inverse, with its two branches given by

$$(2.13) \qquad x \quad = \quad y \pm \sqrt{y^2 + 1}.$$

Therefore, we must split the evaluation of the original integral at the singularity $x = 0$ of $R_2(x)$. The identity (2.10) is written as

$$\begin{aligned} I \quad &= \quad \int_{-\infty}^{0} \frac{C(x)}{E(x)}\, dx + \int_{0}^{\infty} \frac{C(x)}{E(x)}\, dx \\ &= \quad \int_{-\infty}^{\infty} \frac{N_-(y)}{E_1(y)}\, dy + \int_{-\infty}^{\infty} \frac{N_+(y)}{E_1(y)}\, dy \end{aligned}$$

where

$$(2.14) \qquad E_1(y) \quad = \quad \sum_{l=0}^{4} e_l y^{4-l}$$

and

$$(2.15) \qquad N_\pm(y) \quad = \quad \frac{C(y \pm \sqrt{y^2+1})}{Q_2^4(y \pm \sqrt{y^2+1})} \frac{d}{dy}\left(y \pm \sqrt{y^2+1}\right).$$

The new integrand, $(N_+(y)+N_-(y))/E_1(y)$, corresponds to the expression in (1.16). A direct calculation shows that

$$(2.16) \qquad N_-(y) + N_+(y) \quad = \quad 4(202y^2 + 45y + 97),$$

so that

$$(2.17) \qquad I \quad = \quad \int_{-\infty}^{\infty} \frac{202y^2 + 45y + 97}{400y^4 + 1080y^3 + 2969y^2 + 3024y + 3136}\, dy$$
$$= \quad \int_{-\infty}^{\infty} \frac{202y^2 + 45y + 97}{(20y^2 + 27y + 56)^2}\, dy,$$

as claimed.

A proof of a transformation of this type for a general rational integrand is provided in the next four sections.

## 3. A FAMILY OF POLYNOMIALS

For $m \in \mathbb{N}$, we introduce the polynomials

$$(3.1) \qquad P_m(x) \quad = \quad \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m}{2j} x^{m-2j} \quad \text{and}$$

$$(3.2) \qquad Q_m(x) \quad = \quad \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m}{2j+1} x^{m-(2j+1)},$$

which play a fundamental role in the algorithm discussed here. They will comprise the numerators and denominators of a natural change of variables discussed in the last two sections.

The degrees of $P_m$ and $Q_m$ are $m$ and $m-1$, respectively. Observe that

$$(3.3) \qquad P_2(x) = x^2 - 1 \quad \text{and} \quad Q_2(x) = 2x$$

have appeared in Section 2.

**Proposition 3.1.** Let $M(x) = \frac{x+i}{x-i}$ and $f_m(x) = x^m$. Then the rational function $R_m = P_m/Q_m$ satisfies

$$(3.4) \qquad R_m \quad = \quad M^{-1} \circ f_m \circ M.$$

*Proof.* The identity follows from

$$(x+i)^m + (x-i)^m = 2P_m(x) \quad \text{and} \quad (x+i)^m - (x-i)^m = 2iQ_m(x).$$

$\square$

**Corollary 3.2.** The function $R_m$ satisfies

(3.5) $$R_m(\cot\theta) = \cot(m\theta).$$

*Proof.* Use $M(\cot\theta) = e^{2i\theta}$ in (3.4). $\qquad\square$

**Note 3.3.** The multiplicative property $R_n \circ R_m = R_{nm}$ shows that the functions $R_m$ form a family of *commuting* rational functions. The cotangent function in (3.5) appears as the limiting case of the Weierstrass elliptic $\mathfrak{p}$-function,

(3.6) $$\mathfrak{p}(x) = \frac{1}{x^2} + \sum_{n_1, n_2 \in \mathbb{Z}} \left[ \frac{1}{(x - n_1\omega_1 - n_2\omega_2)^2} - \frac{1}{(n_1\omega_1 + n_2\omega_2)^2} \right],$$

where the term $n_1 = n_2 = 0$ is excluded from the sum. In the case $\omega_1 = 1$ and $\omega_2 \to \infty$, we get

(3.7) $$\mathfrak{p}(x) \to -\pi \frac{d}{dx} \cot(\pi x) - \frac{\pi^2}{3}.$$

The function $\mathfrak{p}(nx)$ is even and elliptic, therefore it is a rational function $g_n$ of $\mathfrak{p}$. In view of $g_n \circ g_m = g_{nm}$, these functions commute. An extraordinary fact, due to Ritt [20], is that these are all such commuting rational maps. The functions $R_n$ are a special class of the $g_n$. See [18], section 2.13, for details.

The identity (3.4) permits the explicit evaluation of the zeros of $P_m$ and $Q_m$.

**Proposition 3.4.** The polynomials $P_m$ and $Q_m$ have simple real zeros. Those of $P_m$ are given by

$$p_k = \cot\left( \frac{(2k+1)\pi}{2m} \right) \quad \text{for } 0 \le k \le m - 1,$$

and those of $Q_m$ are

$$q_k = \cot\left( \frac{k\pi}{m} \right) \quad \text{for } 1 \le k \le m - 1.$$

*Proof.* The identity $R_m = M^{-1} \circ f_m \circ M$ yields

$$\begin{aligned} R_m(q_k) &= M^{-1} f_m(M(\cot(k\pi/m))) \\ &= M^{-1}(f_m(e^{2k\pi i/m})) \\ &= M^{-1}(1) = \infty, \end{aligned}$$

so that $Q_m(q_k) = 0$. The degree of $Q_m$ is $m - 1$ and the $q_k$ are all distinct, hence these are all the zeros. The argument for $p_k$ is similar. $\qquad\square$

The polynomials

(3.8) $$P_m^*(a) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m}{2i} x^{2i} \quad \text{and}$$

(3.9) $$Q_m^*(a) = \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} (-1)^i \binom{m}{2i+1} x^{2i+1}$$

have appeared in our development of definite integrals related to the Hurwitz zeta function. See [3] for details. They are connected to $P_m$ and $Q_m$ via

(3.10) $$P_m^*(x) = x^m P_m(x^{-1}) \quad \text{and} \quad Q_m^*(x) = x^{m-1} Q_m(x^{-1}).$$

The role of these polynomials in the development of the Landen transformation comes from their trigonometric properties.

**Proposition 3.5.** The polynomials $P_m^*$ and $Q_m^*$ satisfy

$$(3.11) \qquad P_m^*(\tan\theta) = \frac{\cos m\theta}{\cos^m\theta} \quad \text{and} \quad Q_m^*(\tan\theta) = \frac{\sin m\theta}{\cos^m\theta}.$$

*Proof.* We give the details for $Q_m^*$. The series expansion

$$(3.12) \qquad \frac{\sin(x\tan^{-1}t)}{(1+t^2)^{x/2}} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)_{2k+1}}{(2k+1)!} t^{2k+1},$$

where $(x)_k = x(x+1)(x+2)\cdots(x+k-1)$ is the Pochhammer symbol, is established by checking that both sides satisfy the equation

$$(3.13) \qquad (1+t^2)\frac{d^2g}{dt^2} + 2t(x+1)\frac{dg}{dt} + x(x+1)g = 0,$$

with the initial conditions $g(0) = 0$, $g'(0) = x$. Then $(x)_k$ reduces to

$$(3.14) \qquad (-m)_n = (-1)^n n! \binom{m}{n}$$

for $n \le m$, and vanishes for $n > m$, since $m$ is an integer. Thus (3.12) reduces to

$$\sin(m\tan^{-1}t) = (1+t^2)^{-m/2} \sum_{k=0}^{\lfloor(m-1)/2\rfloor} (-1)^k \binom{m}{2k+1} t^{2k+1}$$

for $x = -m$. This is equivalent to the second formula in (3.11). A similar argument establishes the expression for $P_m^*$. $\qquad\square$

In terms of the original polynomials, (3.11) becomes

$$(3.15) \qquad P_m(\cot\theta) = \frac{\cos m\theta}{\sin^m\theta} \quad \text{and} \quad Q_m(\cot\theta) = \frac{\sin m\theta}{\sin^m\theta}.$$

## 4. A TRIGONOMETRIC REDUCTION

The example described in Section 2 can be extended by using the transformation $y = R_m(x)$ with higher values of $m$. The explicit evaluation of the new integrals requires knowledge of the branches of the inverse map $x = R_m^{-1}(y)$. This is impractical for $m \ge 3$. An alternative method is described in the next section.

The explicit formula for the Landen transformation uses an expression of $\sin^a\theta\cos^b\theta$, for $a$, $b \in \mathbb{N}$, as a linear combination of trigonometric functions of multiple angles.

We introduce the notation

$$(4.1) \qquad c = \lceil \tfrac{a+b}{2} \rceil \text{ and } d = \lfloor \tfrac{a}{2} \rfloor.$$

The reduction formulas given below are expressed in terms of the function

$$(4.2) \qquad T_x(a,b) = \sum_{j=0}^{x} (-1)^{a-x+j} \binom{a}{x-j}\binom{b}{j}.$$

Some of the identities presented here can be found in the table appearing in [12], page 30.

**Proposition 4.1.** Let $a, b \in \mathbb{N}$ and $u \in \mathbb{R}$. Then $\sin^a u \cos^b u$ is given by

$$\frac{(-1)^d}{2^{a+b}} \left[ T_c(a,b) + \sum_{j=1}^{c} (T_{c+j}(a,b) + T_{c-j}(a,b)) \cos(2ju) \right] \quad \text{for } a \text{ even and } b \text{ even,}$$

$$\frac{(-1)^d}{2^{a+b}} \left[ \sum_{j=1}^{c} (T_{c-1+j}(a,b) + T_{c-j}(a,b)) \cos((2j-1)u) \right] \quad \text{for } a \text{ even and } b \text{ odd,}$$

$$\frac{(-1)^d}{2^{a+b}} \left[ \sum_{j=1}^{c} (T_{c-1+j}(a,b) - T_{c-j}(a,b)) \sin((2j-1)u) \right] \quad \text{for } a \text{ odd and } b \text{ even,}$$

$$\frac{(-1)^d}{2^{a+b}} \left[ \sum_{j=1}^{c} (T_{c+j}(a,b) - T_{c-j}(a,b)) \sin(2ju) \right] \quad \text{for } a \text{ odd and } b \text{ odd.}$$

*Proof.* Start with

$$(e^{iu} - e^{-iu})^a (e^{iu} + e^{-iu})^b = \left( \sum_{k=0}^{a} \binom{a}{k}(-1)^{a-k} e^{iu(2k-a)} \right) \left( \sum_{j=0}^{b} \binom{b}{k} e^{iu(2j-b)} \right)$$

$$= \sum_{k=0}^{a} \sum_{j=0}^{b} (-1)^{a-k} \binom{a}{k} \binom{b}{j} e^{iu[2(k+j)-(a+b)]}.$$

Therefore

$$(4.3) \quad \sin^a u \cos^b u = \frac{i^{-a}}{2^{a+b}} \sum_{k=0}^{a+b} \sum_{\nu=0}^{a+b} (-1)^{a-\nu+k} \binom{a}{\nu-k} \binom{b}{k} e^{iu(2\nu-a-b)}.$$

The result follows now by eliminating the imaginary terms on the right hand side of (4.3). $\square$

## 5. The scaling of the integrand

In this section we describe a construction of the polynomials $Z(x)$ and $E(x)$, introduced in Section 2. These are used to produce an appropriate scaling of the integrand in

$$(5.1) \qquad\qquad I = \int_{-\infty}^{\infty} \frac{B(x)}{A(x)} \, dx,$$

so that the new denominator is $E(x)$. Recall that $E$ is the homogeneous polynomial in the variables $(P_m(x), Q_m(x))$ introduced in Section 3.

We now express the coefficients of $E$ and $Z$ in terms of those of $A$. This requires the elementary symmetric functions

$$\sigma_l^{(p)} = \sigma_l^{(p)}(y_1, \cdots, y_p)$$

of the $p$ variables $y_1, \cdots, y_p$. These are defined by the identity

$$(5.2) \qquad\qquad \prod_{l=1}^{p}(y - y_l) = \sum_{l=0}^{p} (-1)^l \sigma_l^{(p)}(y_1, \cdots, y_p) y^{p-l}.$$

**Theorem 5.1.** Let $p, m \in \mathbb{N}$, and

$$(5.3) \qquad A(x) \;=\; \sum_{k=0}^{p} a_k x^{p-k}$$

be a polynomial with real coefficients. Then there exist $mp + 1$ coefficients,

$$(5.4) \qquad z_0, \, z_1, \cdots, z_r; \; e_1, \, e_2, \cdots, e_p,$$

with $r = p(m-1)$, such that

$$(5.5) \qquad A(x) Z(x) \;=\; E(x),$$

where

$$(5.6) \qquad Z(x) := \sum_{k=0}^{r} z_k x^{r-k} \quad \text{and} \quad E(x) := \sum_{l=0}^{p} e_l \left[ P_m(x) \right]^{p-l} \left[ Q_m(x) \right]^{l}.$$

The coefficients $e_l$ are polynomials in the coefficients $\frac{a_1}{a_0}, \cdots, \frac{a_p}{a_0}$.

**Note 5.2.** The effect of the theorem is to scale the integrand $A(x)/B(x)$ to $C(x)/E(x)$, where $C(x) = B(x)Z(x)$ and $E(x) = A(x)Z(x)$. The degrees are recorded here:

$$(5.7) \qquad \deg(A) = p, \, \deg(B) = p - 2, \, \deg(Z) = r = pm - p,$$
$$\deg(C) = s = pm - 2, \text{ and } \deg(E) = pm.$$

*Proof.* Let $\{x_1, x_2, \ldots, x_p\}$ be the roots of $A$, each written according to its multiplicity, so that

$$(5.8) \qquad A(x) = a_0 \prod_{j=1}^{p} (x - x_j).$$

The rational function

$$(5.9) \qquad R_m(x) = \frac{P_m(x)}{Q_m(x)},$$

introduced in Proposition 3.1, is well-defined at all the roots $x_j$. This follows from the fact that the roots of $Q_m$ are real and our assumption that the roots $x_1, \cdots, x_p$ of $A(x) = 0$ are not. For $0 \le l \le p$, define

$$(5.10) \qquad e_l := a_0^m (-1)^l \prod_{j=1}^{p} Q_m(x_j) \times \sigma_l^{(p)} \left( R_m(x_1), \, R_m(x_2), \ldots, R_m(x_p) \right)$$

and the polynomial

$$(5.11) \qquad H(x) = \sum_{l=0}^{p} e_l x^{p-l}.$$

We now consider the identity,

$$(5.12) \qquad \prod_{j=1}^{p} (y - R_m(x_j)) = \sum_{l=0}^{p} (-1)^l \sigma_l^{(p)} \left( R_m(x_1), \ldots, R_m(x_p) \right) y^{p-l},$$

that comes from (5.2). Clearing denominators, we obtain

$$(5.13) \qquad \prod_{j=1}^{p}(Q_m(x_j)y - P_m(x_j)) = a_0^{-m}\sum_{l=0}^{p}e_l y^{p-l} = a_0^{-m}H(y).$$

In particular,

$$(5.14) \qquad H(R_m(x)) = a_0^m \prod_{j=1}^{p}Q_m(x_j) \times \prod_{j=1}^{p}(R_m(x) - R_m(x_j)).$$

Finally, define the polynomial

$$\begin{aligned} E(x) &= \sum_{l=0}^{p}e_l P_m^{p-l}(x)Q_m^l(x) \\ &= H(R_m(x))Q_m^p(x). \end{aligned}$$

Identity (5.14) shows that the zeros of $E$ are precisely the values $R_m(x_j)$, $1 \le j \le p$. The coefficients of $E$, given in (5.13), are symmetric polynomials of the roots $x_j$ of $A$. The fundamental theorem of symmetric polynomials [2] states that $e_l$ is a polynomial in $\frac{a_1}{a_0}, \cdots, \frac{a_p}{a_0}$. This, in turn, proves that $e_l \in \mathbb{R}$ and thus $E \in \mathbb{R}[x]$.

Now observe that (5.14) yields $E(x_j) = 0$ and the corresponding factor $R_m(x) - R_m(x_j)$ appears with the same multiplicity as $x_j$. We conclude that $A$ divides $E$ and define $Z$ to be the quotient.

$\square$

## 6. The reduction of the integrand

In this section, we produce explicit formulas for rational Landen transformations of the integral

$$(6.1) \qquad I = \int_{-\infty}^{\infty}\frac{B(x)}{A(x)}\,dx.$$

As before, we assume that $A$, $B \in \mathbb{R}[x]$, relatively prime, and that $I < \infty$.

In Section 5 we have scaled the integrand in (6.1) to the form

$$(6.2) \qquad I = \int_{-\infty}^{\infty}\frac{C(x)}{E(x)}\,dx,$$

where the denominator is written as

$$(6.3) \qquad E(x) = \sum_{l=0}^{p}e_l P_m^{p-l}(x)Q_m^l(x).$$

Here, $P_m$ and $Q_m$ are the polynomials discussed in Section 3. The scaling of the denominator is achieved through multiplication by $Z(x)$, as given in Theorem 5.1.

The numerator becomes

$$(6.4) \qquad C(x) = B(x)Z(x) = \sum_{k=0}^{s}c_k x^{s-k}.$$

The coefficients $c_k$ are given by

$$(6.5) \qquad c_j = \sum_{k=0}^{j}z_k b_{j-k} \quad \text{for } 0 \le j \le s,$$

where $b_i = 0$ if $i > p - 2$ and $z_i = 0$ if $i > r = pm - p$.

The change of variables $x = \cot\theta$ and the relations (3.15) yield

$$(6.6) \qquad I \;=\; \int_0^\pi \frac{\mathrm{CT}_{m,p}(\theta)}{\mathrm{ET}_{m,p}(\theta)}\,d\theta,$$

where

$$(6.7) \qquad \mathrm{CT}_{m,p}(\theta) \;=\; \sum_{k=0}^{s} c_k \cos^{s-k}\theta \, \sin^k\theta$$

and

$$(6.8) \qquad \mathrm{ET}_{m,p}(\theta) \;=\; \sum_{l=0}^{p} e_l \cos^{p-l}(m\theta) \, \sin^l(m\theta).$$

The discussion of this integral is divided according to the parity of $m$. Recall that $p$ is assumed to be even. The details are presented in the case $m$ odd.

The parameter $s = mp - 2$ is even and we write $s = 2\lambda$. Split (6.6) as

$$
\begin{aligned}
I \;&=\; \sum_{j=0}^{\lambda} c_{2j} \int_0^\pi \frac{\sin^{2j}\theta \, \cos^{s-2j}\theta \, d\theta}{\mathrm{ET}_{m,p}(\theta)} + \sum_{j=0}^{\lambda-1} c_{2j+1}\int_0^\pi \frac{\sin^{2j+1}\theta \, \cos^{s-2j-1}\theta \, d\theta}{\mathrm{ET}_{m,p}(\theta)} \\
&\equiv\; I_1 + I_2,
\end{aligned}
$$

and consider the evaluation of each of these integrals.

*The evaluation of $I_1$.* The identity in Proposition 4.1 yields

$$
\begin{aligned}
\sin^{2j}\theta \, \cos^{s-2j}\theta \;=\;& \frac{(-1)^j}{2^s} T_\lambda(2j, s - 2j) + \\
& +\; \frac{(-1)^j}{2^s} \sum_{k=1}^{\lambda} [T_{\lambda+k}(2j, s-2j) + T_{\lambda-k}(2j, s-2j)]\cos(2k\theta),
\end{aligned}
$$

and replacing this in the definition of $I_1$ yields

(6.9)

$$
\begin{aligned}
I_1 \;=\;& \frac{1}{2^s}\sum_{j=0}^{\lambda}(-1)^j c_{2j} T_\lambda(2j, s-2j)\int_0^\pi \frac{d\theta}{\mathrm{ET}_{m,p}(\theta)} + \\
& +\; \frac{1}{2^s}\sum_{j=0}^{\lambda}(-1)^j c_{2j}\sum_{k=1}^{\lambda}[T_{\lambda+k}(2j, s-2j) + T_{\lambda-k}(2j, s-2j)]\int_0^\pi \frac{\cos(2k\theta)\,d\theta}{\mathrm{ET}_{m,p}(\theta)}.
\end{aligned}
$$

The periodicity of the integrand, and the fact that $mp$ is even, show that the integrals appearing in (6.9) are half of the corresponding ones over the whole period $[0, 2\pi]$. Thus,

(6.10)

$$
\begin{aligned}
I_1 \;=\;& \frac{1}{2^{s+1}}\sum_{j=0}^{\lambda}(-1)^j c_{2j} T_\lambda(2j, s-2j)\int_0^{2\pi} \frac{d\theta}{\mathrm{ET}_{m,p}(\theta)} + \\
& +\; \frac{1}{2^{s+1}}\sum_{j=1}^{\lambda}(-1)^j c_{2j}\sum_{k=1}^{\lambda}[T_{\lambda+k}(2j, s-2j) + T_{\lambda-k}(2j, s-2j)]\int_0^{2\pi} \frac{\cos(2k\theta)\,d\theta}{\mathrm{ET}_{m,p}(\theta)}.
\end{aligned}
$$

We now show that most of the integrals in (6.10) vanish. This reduction is responsible for the existence of a rational Landen transformation.

Introduce the notation

$$(6.11) \qquad S_{m,p}(k) \quad := \quad \int_0^{2\pi} \frac{\sin(k\theta)\, d\theta}{\mathrm{ET}_{m,p}(\theta)}$$

and

$$(6.12) \qquad C_{m,p}(k) \quad := \quad \int_0^{2\pi} \frac{\cos(k\theta)\, d\theta}{\mathrm{ET}_{m,p}(\theta)}.$$

**Lemma 6.1.** Let $k, m, p \in \mathbb{N}$ be arbitrary. Then $S_{m,p}(k)$ and $C_{m,p}(k)$ vanish unless $k$ is a multiple of $m$.

*Proof.* In the definition of $S_{m,p}(k)$ let $\theta \mapsto \theta + 2\pi j/m$ for $j = 0, 1, \cdots, m-1$. The average of these $m$ integrals is

$$S_{m,p}(k) \quad = \quad \frac{1}{m} \int_0^{2\pi} \frac{d\theta}{\mathrm{ET}_{m,p}(\theta)} \left( \sin(k\theta) \sum_{j=0}^{m-1} \cos(2\pi k j/m) + \cos(k\theta) \sum_{j=0}^{m-1} \sin(2\pi k j/m) \right).$$

If $k$ is not a multiple of $m$ the integrand vanishes because the sums in it are the real and imaginary parts of

$$\sum_{j=0}^{m-1} e^{2\pi i j k/m} \quad = \quad \frac{1 - e^{2\pi i k}}{1 - e^{2\pi i k/m}} = 0.$$

A similar proof follows for $C_{m,p}(k)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In view of Lemma 6.1, we replace $k$ by $\alpha m$, where $1 \leq \alpha \leq \nu - 1$, with $\nu = p/2$. Then (6.10) becomes

$$(6.13)$$
$$I_1 \quad = \quad \frac{1}{2^{s+1}} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \int_0^{2\pi} \frac{d\theta}{\mathrm{ET}_{m,p}(\theta)} \; +$$
$$+ \quad \frac{1}{2^{s+1}} \sum_{j=1}^{\lambda} (-1)^j c_{2j} \sum_{\alpha=1}^{\nu-1} [T_{\lambda+\alpha m}(2j, s-2j) + T_{\lambda-\alpha m}(2j, s-2j)] \int_0^{2\pi} \frac{\cos(2\alpha m\theta)\, d\theta}{\mathrm{ET}_{m,p}(\theta)}.$$

The change of variables $\varphi = m\theta$ produces

$$(6.14) \qquad \int_0^{2\pi} \cdots d\theta = \frac{1}{m} \int_0^{2\pi m} \cdots d\varphi = \int_0^{2\pi} \cdots d\varphi,$$

using the periodicity of the integrand. We conclude that

(6.15)

$$I_1 \;\; = \;\; \frac{1}{2^{s+1}} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \int_0^{2\pi} \frac{d\theta}{\mathrm{ET}_{1,p}(\theta)} \; +$$

$$+ \;\; \frac{1}{2^{s+1}} \sum_{j=1}^{\lambda} (-1)^j c_{2j} \sum_{\alpha=1}^{\nu-1} [T_{\lambda+\alpha m}(2j, s-2j) + T_{\lambda-\alpha m}(2j, s-2j)] \int_0^{2\pi} \frac{\cos(2\alpha\theta)\, d\theta}{\mathrm{ET}_{1,p}(\theta)},$$

where the denominator is

(6.16) $$\mathrm{ET}_{1,p}(\theta) \;\; = \;\; \sum_{l=0}^{p} e_l \cos^{p-l}\theta \sin^l\theta.$$

The next step is to bring back the domain of integration to $[0, \pi]$. The symmetry of the integrand shows that the integral over $[\pi, 2\pi]$ is the same as that over $[0, \pi]$. We conclude that

(6.17)

$$I_1 \;\; = \;\; \frac{1}{2^s} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \int_0^{\pi} \frac{d\theta}{\mathrm{ET}_{1,p}(\theta)} \; +$$

$$+ \;\; \frac{1}{2^s} \sum_{j=1}^{\lambda} (-1)^j c_{2j} \sum_{\alpha=1}^{\nu-1} [T_{\lambda+\alpha m}(2j, s-2j) + T_{\lambda-\alpha m}(2j, s-2j)] \int_0^{\pi} \frac{\cos(2\alpha\theta)\, d\theta}{\mathrm{ET}_{1,p}(\theta)}.$$

The change of variables $y = \cot\theta$ gives, recalling that $\nu = p/2$,

(6.18) $$\int_0^{\pi} \frac{d\theta}{\mathrm{ET}_{1,p}(\theta)} \;\; = \;\; \int_{-\infty}^{\infty} (1+y^2)^{\nu-1} \frac{dy}{H(y)} \;,$$

where the polynomial

(6.19) $$H(y) \;\; = \;\; \sum_{l=0}^{p} e_l y^{p-l}$$

was introduced in (5.11). The identity (3.15) is now used to change variables in the second integral to obtain

(6.20) $$\int_0^{\pi} \frac{\cos(2\alpha\theta)\, d\theta}{\mathrm{ET}_{1,p}(\theta)} \;\; = \;\; \int_{-\infty}^{\infty} (1+y^2)^{\nu-\alpha-1} P_{2\alpha}(y) \frac{dy}{H(y)}.$$

The next step is to write $P_{2\alpha}(y)$ in terms of $1+y^2$.

**Lemma 6.2.** The polynomial $P_{2\alpha}(y)$ can be written as

(6.21) $$P_{2\alpha}(y) \;\; = \;\; \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \frac{\alpha}{2\alpha-\beta} 2^{2(\alpha-\beta)} \binom{2\alpha-\beta}{\beta} (1+y^2)^\beta.$$

*Proof.* Start with

$$
\begin{aligned}
P_{2\alpha}(x) &= \sum_{j=0}^{\alpha} (-1)^j \binom{2\alpha}{2j} x^{2\alpha-2j} \\
&= \sum_{j=0}^{\alpha} (-1)^j \binom{2\alpha}{2j} \left[ (1+x^2) - 1 \right]^{\alpha-j} \\
&= \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \left( \sum_{j=0}^{\alpha-\beta} \binom{2\alpha}{2j} \binom{\alpha-j}{\beta} \right) (1+x^2)^{\beta},
\end{aligned}
$$

and the result follows from

$$
(6.22) \qquad \sum_{j=0}^{\alpha-\beta} \binom{2\alpha}{2j} \binom{\alpha-j}{\beta} = \frac{\alpha}{2\alpha-\beta} \binom{2\alpha-\beta}{\beta} 2^{2(\alpha-\beta)}, \text{ for } \alpha \geq \beta.
$$

This sum arises as a corollary of Gauss's hypergeometric evaluation [10],

$$
(6.23) \qquad {}_2F_1 \left[ a, b; c; 1 \right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},
$$

valid for $\operatorname{Re}(c-a-b) > 0$. In our case, $a = \frac{1}{2} - \alpha$, $b = \beta - \alpha$ and $c = \frac{1}{2}$, so that $c - a - b = 2\alpha - \beta > 0$. See [1], page 66 for a proof of (6.23). $\qquad \square$

We return to the evaluation of $I_1$. The expression in (6.17) becomes

$$
\begin{aligned}
I_1 &= \frac{1}{2^s} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \int_{-\infty}^{\infty} (1+y^2)^{\nu-1} \frac{dy}{H(y)} + \\
&\quad + \frac{1}{2^s} \sum_{j=0}^{\lambda} (-1)^j c_{2j} \sum_{\alpha=1}^{\nu-1} \left( T_{\lambda+\alpha m}(2j-s-2j) + T_{\lambda-\alpha m}(2j-s-2j) \right) \times \\
&\quad \times \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} 2^{2(\alpha-\beta)} \frac{\alpha}{2\alpha-\beta} \binom{2\alpha-\beta}{\beta} \int_{-\infty}^{\infty} (1+y^2)^{\nu-\alpha-1+\beta} \frac{dy}{H(y)}.
\end{aligned}
$$

Expanding the powers of $1 + y^2$, and reversing the order of summation, leads to

$$
\begin{aligned}
I_1 &= \frac{1}{2^s} \sum_{\gamma=0}^{\nu-1} \binom{\nu-1}{\gamma} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \int_{-\infty}^{\infty} y^{2\gamma} \frac{dy}{H(y)} + \\
&\quad + \frac{1}{2^s} \sum_{\gamma=0}^{\nu-2} \left( \sum_{j=0}^{\lambda} \sum_{\alpha=1}^{\nu-1-\gamma} \sum_{\beta=0}^{\alpha} M_1(j, \alpha, \beta; m, p) \right) \int_{-\infty}^{\infty} y^{2\gamma} \frac{dy}{H(y)} + \\
&\quad + \frac{1}{2^s} \sum_{\gamma=1}^{\nu-1} \left( \sum_{j=0}^{\lambda} \sum_{\alpha=\nu-\gamma}^{\nu-1} \sum_{\beta=\alpha-\nu+\gamma+1}^{\alpha} M_1(j, \alpha, \beta; m, p) \right) \int_{-\infty}^{\infty} y^{2\gamma} \frac{dy}{H(y)},
\end{aligned}
$$

where

$$
\begin{aligned}
M_1(j, \alpha, \beta; m, p) \quad = \quad & (-1)^{j+\alpha-\beta} c_{2j} \frac{2^{2(\alpha-\beta)} \alpha}{2\alpha - \beta} \binom{2\alpha - \beta}{\beta} \binom{\nu - \alpha - 1 + \beta}{\gamma} \\
\times \quad & [T_{\lambda+\alpha m}(2j, s - 2j) + T_{\lambda-\alpha m}(2j, s - 2j)].
\end{aligned}
$$

*The evaluation of $I_2$.* A similar calculation leads to

$$
\begin{aligned}
I_2 \quad = \quad & \frac{1}{2^s} \sum_{\gamma=0}^{\nu-2} \left( \sum_{j=0}^{\lambda-1} \sum_{\alpha=1}^{\nu-1-\gamma} \sum_{\beta=0}^{\alpha-1} M_2(j, \alpha, \beta; m, p) \right) \int_{-\infty}^{\infty} y^{2\gamma+1} \frac{dy}{H(y)} + \\
+ \quad & \frac{1}{2^s} \sum_{\gamma=1}^{\nu-2} \left( \sum_{j=0}^{\lambda-1} \sum_{\alpha=\nu-\gamma}^{\nu-1} \sum_{\beta=0}^{\alpha-1} M_2(j, \alpha, \beta; m, p) \right) \int_{-\infty}^{\infty} y^{2\gamma+1} \frac{dy}{H(y)},
\end{aligned}
$$

where

$$
\begin{aligned}
M_2(j, \alpha, \beta; m, p) \quad = \quad & (-1)^{j+\beta} c_{2j+1} 2^{2\beta+1} \binom{\alpha + \beta}{2\beta + 1} \binom{\nu - 2 - \beta}{\gamma} \\
\times \quad & [T_{\lambda+\alpha m}(2j + 1, s - 2j - 1) - T_{\lambda-\alpha m}(2j + 1, s - 2j - 1)].
\end{aligned}
$$

For the convenience of the reader we summarize the information as a theorem.

**Theorem 6.3.** Let $p, m \in \mathbb{N}$ and assume $p$ is even and $m$ is odd. Define

$$
\text{(6.24)} \qquad s = mp - 2, \ r = p(m - 1), \ \lambda = \frac{s}{2}, \ \text{and} \ \nu = \frac{p}{2},
$$

and consider the polynomials

$$
\text{(6.25)} \qquad A(x) = \sum_{k=0}^{p} a_k x^{p-k} \quad \text{and} \quad B(x) = \sum_{k=0}^{p-2} b_k x^{p-2-k}.
$$

Then

$$
\text{(6.26)} \qquad \int_{-\infty}^{\infty} \frac{B(x)}{A(x)} \, dx \quad = \quad \int_{-\infty}^{\infty} \frac{J(x)}{H(x)} \, dx.
$$

The new denominator $H$ is given by

$$
\text{(6.27)} \qquad H(x) \quad = \quad \sum_{l=0}^{p} e_l x^{p-l},
$$

where, for $0 \leq j \leq p$, the coefficients $e_j$ are solutions to the system (5.5). An expression for $e_j$ in terms of the coefficients $a_k$ is given in (5.10).

The new numerator $J$ employs the function

$$
\text{(6.28)} \qquad T_x(a, b) \quad = \quad \sum_{j=0}^{x} (-1)^{a-x+j} \binom{a}{x - j} \binom{b}{j},
$$

and is given by

$$
(6.29) \quad J(x) = \frac{1}{2^s} \sum_{\gamma=0}^{\nu-1} \left( \binom{\nu-1}{\gamma} \sum_{j=0}^{\lambda} (-1)^j c_{2j} T_\lambda(2j, s-2j) \right) x^{2\gamma}
$$

$$
+ \quad \frac{1}{2^s} \sum_{\gamma=0}^{\nu-2} \left( \sum_{j=0}^{\lambda} \sum_{\alpha=1}^{\nu-1-\gamma} \sum_{\beta=0}^{\alpha} M_1(j, \alpha, \beta; m, p) \right) x^{2\gamma}
$$

$$
+ \quad \frac{1}{2^s} \sum_{\gamma=1}^{\nu-1} \left( \sum_{j=0}^{\lambda} \sum_{\alpha=\nu-\gamma}^{\nu-1} \sum_{\beta=\alpha-\nu+\gamma+1}^{\alpha} M_1(j, \alpha, \beta; m, p) \right) x^{2\gamma}
$$

$$
+ \quad \frac{1}{2^s} \sum_{\gamma=0}^{\nu-2} \left( \sum_{j=0}^{\lambda-1} \sum_{\alpha=1}^{\nu-1-\gamma} \sum_{\beta=0}^{\alpha-1} M_2(j, \alpha, \beta; m, p) \right) x^{2\gamma+1}
$$

$$
+ \quad \frac{1}{2^s} \sum_{\gamma=1}^{\nu-2} \left( \sum_{j=0}^{\lambda-1} \sum_{\alpha=\nu-\gamma}^{\nu-1} \sum_{\beta=0}^{\alpha-1} M_2(j, \alpha, \beta; m, p) \right) x^{2\gamma+1}.
$$

The coefficients $c_j$ are given by

$$
(6.30) \qquad\qquad c_j = \sum_{k=0}^{j} z_k b_{j-k} \quad \text{for } 0 \le j \le s,
$$

with $b_i = 0$ if $i > p - 2$ and $z_i = 0$ if $i > r = mp - p$. The values of $z_j$ are obtained as solutions of the system (5.5).

Finally,

$$
M_1(j, \alpha, \beta; m, p) = (-1)^{j+\alpha-\beta} c_{2j} \frac{2^{2(\alpha-\beta)} \alpha}{2\alpha - \beta} \binom{2\alpha - \beta}{\beta} \binom{\nu - \alpha - 1 + \beta}{\gamma}
$$
$$
\times \quad [T_{\lambda+\alpha m}(2j, s-2j) + T_{\lambda-\alpha m}(2j, s-2j)],
$$

and

$$
M_2(j, \alpha, \beta; m, p) = (-1)^{j+\beta} c_{2j+1} 2^{2\beta+1} \binom{\alpha + \beta}{2\beta + 1} \binom{\nu - 2 - \beta}{\gamma}
$$
$$
\times \quad [T_{\lambda+\alpha m}(2j+1, s-2j-1) - T_{\lambda-\alpha m}(2j+1, s-2j-1)].
$$

**Note 6.4.** Surprisingly, the expressions for $H$ and $J$ given in (6.27) and (6.29) remain valid when $m$ is even. This results from a similar calculation whose details are omitted here.

## 7. EXAMPLES OF RATIONAL LANDEN TRANSFORMATIONS

This section contains some examples that illustrate the rational Landen transformations.

**Example 7.1.** We calculate the transformation for the case $p = m = 2$. The integrand in

$$(7.1) \qquad I \; = \; \int_{-\infty}^{\infty} \frac{dx}{a_0 x^2 + a_1 x + a_2}, \; \text{ with } a_0 \neq 0 \;,$$

is quadratic, thus $p = 2$. We construct the Landen transformation with convergence order $m = 2$. Therefore, $s = mp - 2 = 2$ and $r = p(m - 1) = 2$ in the notation defined in Theorem 5.1. The scaling of Section 5 amounts to finding parameters $z_0, z_1, z_2$, and $e_0, e_1, e_2$ such that

$$(a_0 x^2 + a_1 x + a_2)(z_0 x^2 + z_1 x + z_2) \;\; = \;\; e_0 P_2^2(x) + e_1 P_2(x) Q_2(x) + e_2 Q_2^2(x),$$

with $P_2(x) = x^2 - 1$ and $Q_2(x) = 2x$. The linear system (5.5) is of order $mp + 1 = 5$ and we choose the free parameter

$$(7.2) \qquad e_0 \;\; = \;\; Q_2(x_1) Q_2(x_2) = 4x_1 x_2 = \frac{4a_2}{a_0},$$

according to the convention in (5.10). The solution of (5.5) yields

$$(7.3) \qquad z_0 = \frac{4a_2}{a_0^2}, \;\; z_1 = -\frac{4a_1}{a_0^2}, \;\; z_2 = \frac{4}{a_0},$$

and the formulas in (5.10) produce

$$(7.4) \qquad e_0 = \frac{4a_2}{a_0}, \;\; e_1 = \frac{2a_1(a_2 - a_0)}{a_0^2}, \;\; e_2 = \frac{(a_0 + a_2)^2 - a_1^2}{a_0^2}.$$

Therefore, the denominator of the new integrand is

$$(7.5) \qquad H(x) \;\; = \;\; \frac{4a_2}{a_0} x^2 + \frac{2a_1(a_2 - a_0)}{a_0^2} x + \frac{(a_0 + a_2)^2 - a_1^2}{a_0^2}.$$

The new numerator is obtained from the formulas given in Theorem 6.3. In this case $\lambda = 1$ and $\nu = 1$, thus only one sum contributes to its value:

$$(7.6) \qquad J(x) \;\; = \;\; \frac{2(a_0 + a_2)}{a_0^2}.$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{dx}{a_0 x^2 + a_1 x + a_2} \;\; = \;\; \int_{-\infty}^{\infty} \frac{2(a_0 + a_2)\, dx}{4a_0 a_2 x^2 + 2a_1(a_2 - a_0)x + [(a_0 + a_2)^2 - a_1^2]},$$

and (7.1) is invariant under the transformation

$$(7.7) \qquad a_0 \;\; \mapsto \;\; \frac{2a_0 a_2}{a_0 + a_2},$$

$$a_1 \;\; \mapsto \;\; \frac{a_1(a_2 - a_0)}{a_0 + a_2},$$

$$a_2 \;\; \mapsto \;\; \frac{(a_0 + a_2)^2 - a_1^2}{2(a_0 + a_2)}.$$

This was announced in (1.10).

**Example 7.2.** We present the Landen transformation of order 3 for the rational function

$$(7.8) \qquad R(x) \;\; = \;\; \frac{x^2 + 4x + 4}{x^6 + 16x^5 + 114x^4 + 452x^3 + 1041x^2 + 1300x + 676}.$$

This is an example that violates the main assumption on the nature of the roots of $A$. Indeed,

$$(7.9) \qquad A(x) = (x+2)^2(x^2 + 6x + 13)^2 \quad \text{and} \quad B(x) = (x+2)^2,$$

so that $A$ has real roots. Although $R$ is not reduced, it is integrable over $\mathbb{R}$ :

$$I \;\; = \;\; \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 6x + 13)^2} = \frac{\pi}{16}.$$

This example shows that the rational Landen transformations preserve the existence of real poles of the integrand. Moreover, the real zeros that cancel these singularities are transformed accordingly to preserve convergence.

The roots of $A(x) = 0$ are

$$(7.10) \qquad x_1 = x_2 = -3 - 2i, \; x_3 = x_4 = -3 + 2i, \; x_5 = x_6 = -2.$$

The value of $e_0$ given in (5.10) yields

$$(7.11) \qquad\qquad e_0 \;\; = \;\; \prod_{k=1}^{6} Q_3(x_k) = 269353744.$$

We have that $p = 6$ and $m = 3$, and so $r = 12$ and $s = 16$. Solving a system of order 19 yields

$$(7.12) \qquad\qquad A_1(x) \;\; = \;\; (11x + 2)^2(373x^2 + 594x + 481)^2$$

as the new denominator, and the new numerator is

$$(7.13) \qquad\qquad B_1(x) \;\; = \;\; (11x + 2)^2(854x^2 + 3240x + 10709).$$

Observe that the algorithm preserves the existence of a real root, but the root at $x = -2/11$ is cancelled. The reader will check the invariance:

$$(7.14) \qquad\qquad \int_{-\infty}^{\infty} \frac{854x^2 + 3240x + 10709}{(373x^2 + 594x + 481)^2}\, dx \;\; = \;\; \frac{\pi}{16} \;.$$

**Example 7.3.** Finally we present a numerical example that illustrates the convergence of the iterative transformations constructed in this paper. The original integral is written in the form

$$(7.15) \quad I \;\; = \;\; \frac{b_0}{a_0} \int_{-\infty}^{\infty} \frac{x^{p-2} + b_0^{-1}b_1 x^{p-3} + b_0^{-1}b_2 x^{p-4} + \cdots + b_0^{-1}b_{p-2}}{x^p + a_0^{-1}a_1 x^{p-1} + a_0^{-1}a_2 x^{p-2} + \cdots + a_0^{-1}a_p}\, dx.$$

The Landen transformation generates a sequence of coefficients,

$$(7.16) \qquad\qquad \mathfrak{P}_n \;\; := \;\; \{a_0^{(n)}, a_1^{(n)}, \cdots, a_p^{(n)}; b_0^{(n)}, b_1^{(n)}, \cdots, b_{p-2}^{(n)}\} \;,$$

with $\mathfrak{P}_0 = \mathfrak{P}$ as in (1.6). We wish to show that, as $n \to \infty$,

$$(7.17) \qquad\qquad u_n := \left( \frac{a_1^{(n)}}{a_0^{(n)}}, \frac{a_2^{(n)}}{a_0^{(n)}}, \cdots, \frac{a_p^{(n)}}{a_0^{(n)}}, \frac{b_1^{(n)}}{b_0^{(n)}}, \frac{b_2^{(n)}}{b_0^{(n)}}, \cdots, \frac{b_{p-2}^{(n)}}{b_0^{(n)}} \right)$$

converges to

$$(7.18) \quad u_\infty := \left(0, \binom{q}{1}, 0, \binom{q}{2}, \cdots, \binom{q}{q}; 0, \binom{q-1}{1}, 0, \binom{q-1}{2}, \cdots, \binom{q-1}{q-1}\right),$$

where $q = p/2$. The invariance of the integral then shows that

$$(7.19) \qquad \qquad \frac{b_0^{(n)}}{a_0^{(n)}} \to \frac{1}{\pi} I.$$

The convergence of $v := u_n - u_\infty$ to $0$ is measured in the $L_2$−norm,

$$(7.20) \qquad \qquad \|v\|_2 \quad = \quad \frac{1}{\sqrt{2p-2}} \left( \sum_{k=1}^{2p-2} \|v_k\|^2 \right)^{1/2},$$

and also the $L_\infty$-norm,

$$(7.21) \qquad \qquad \|v\|_\infty \quad = \quad \text{Max} \left\{ \|v_k\| : 1 \le k \le 2p - 2 \right\}.$$

The rational functions appearing as integrands have rational coefficients, so, as a measure of their complexity, we take the largest number of digits of these coefficients. This appears in the column marked *size*.

The following tables illustrate the iterates of rational Landen transformations of order 2, 3 and 4, applied to the example

$$(7.22) \qquad I \quad = \quad \int_{-\infty}^{\infty} \frac{3x + 5}{x^4 + 14x^3 + 74x^2 + 184x + 208} \, dx = -\frac{7\pi}{12}.$$

The first column gives the $L_2$-norm of $u_n - u_\infty$, the second its $L_\infty$-norm, the third presents the relative error in (7.19), and in the last column we give the size of the rational integrand. At each step, we verify that the new rational function integrates to $-7\pi/12$.

<div align="center">Method of order 2</div>

| $n$ | $L_2$-norm | $L_\infty$-norm | Error | Size |
|---|---|---|---|---|
| 1 | 58.7171 | 69.1000 | 1.02060 | 5 |
| 2 | 7.444927 | 9.64324 | 1.04473 | 10 |
| 3 | 4.04691 | 5.36256 | 0.945481 | 18 |
| 4 | 1.81592 | 2.41858 | 1.15092 | 41 |
| 5 | 0.360422 | 0.411437 | 0.262511 | 82 |
| 6 | 0.0298892 | 0.0249128 | 0.0189903 | 164 |
| 7 | 0.000256824 | 0.000299728 | 0.0000362352 | 327 |
| 8 | $1.92454 \times 10^{-8}$ | $2.24568 \times 10^{-8}$ | $1.47053 \times 10^{-8}$ | 659 |
| 9 | $1.0823 \times 10^{-16}$ | $1.2609 \times 10^{-16}$ | $8.2207 \times 10^{-17}$ | 1318 |

As expected we observe quadratic convergence in the $L_2$−norm and also in the $L_\infty$−norm. The size of the integrand is doubled at each iteration.

Method of order 3

| $n$ | $L_2$-norm | $L_\infty$-norm | Error | Size |
|---|---|---|---|---|
| 1 | 15.2207 | 20.2945 | 1.03511 | 8 |
| 2 | 1.97988 | 1.83067 | 0.859941 | 23 |
| 3 | 0.41100 | 0.338358 | 0.197044 | 69 |
| 4 | 0.00842346 | 0.00815475 | 0.00597363 | 208 |
| 5 | $5.05016 \times 10^{-8}$ | $5.75969 \times 10^{-8}$ | $1.64059 \times 10^{-9}$ | 626 |
| 6 | $1.09651 \times 10^{-23}$ | $1.02510 \times 10^{-23}$ | $3.86286 \times 10^{-24}$ | 1878 |
| 7 | $1.12238 \times 10^{-70}$ | $1.22843 \times 10^{-70}$ | $8.59237 \times 10^{-71}$ | 5634 |

Method of order 4

| $n$ | $L_2$-norm | $L_\infty$-norm | Error | Size |
|---|---|---|---|---|
| 1 | 7.44927 | 9.64324 | 1.04473 | 10 |
| 2 | 1.81592 | 2.41858 | 1.15092 | 41 |
| 3 | 0.0298892 | 0.0249128 | 0.0189903 | 164 |
| 4 | $1.92454 \times 10^{-8}$ | $2.249128 \times 10^{-8}$ | $1.47053 \times 10^{-8}$ | 659 |
| 5 | $3.40769 \times 10^{-33}$ | $3.96407 \times 10^{-33}$ | $2.56817 \times 10^{-33}$ | 2637 |

## 8. Conclusions

We have presented an algorithm for the evaluation of a rational integral over ℝ. Numerical evidence of its convergence is presented.

## References

[1] G. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.

[2] M. Artin. *Algebra*. Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[3] G. Boros, O. Espinosa, and V. Moll. On some families of integrals solvable in terms of polygamma and negapolygamma functions. *Integrals Transforms and Special Functions*, 14:187–203, 2003.

[4] G. Boros and V. Moll. A rational Landen transformation. The case of degree 6. In Knopp G. Mendoza E.T. Quinto E. L. Grinberg S. Berhanu M, editor, *Contemporay Mathematics. Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis*, volume 251, pages 83–89. American Mathematical Society, 2000.

[5] G. Boros and V. Moll. Landen transformation and the integration of rational functions. *Math. Comp.*, 71:649–668, 2001.

[6] J. M. Borwein and P. B. Borwein. *Pi and the AGM- A study in analytic number theory and computational complexity*. Wiley, New York, 1st edition, 1987.

[7] M. Bronstein. *Symbolic Integration I. Transcendental functions*, volume 1 of *Algorithms and Computation in Mathematics*. Springer-Verlag, 1997.

[8] M. Chamberland and V. Moll. Dynamics of the degree six Landen transformation. *Discrete and Dynamical Systems*, 2006.

[9] K. F. Gauss. Arithmetisch Geometrisches Mittel. *Werke*, 3:361–432, 1799.

[10] K. F. Gauss. Disquisitiones generales circa seriem infinitam. *Werke*, 3:123–162, 1812.

[11] K. Geddes and T. C. Scott. Recipes for classes of definite integrals involving exponentials and logarithms. In E. Kaltofen S. M. Watt, editor, *Proceedings of Computers and Mathematics (conference held at MIT*, pages 192–201. Springer, Berlin - Heidelberg - New York, 1989.

[12] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

[13] C. Hermite. Sur l'integration des fractions rationelles. *Nouvelles Annales de Mathematiques (2eme serie)*, 11:145–148, 1872.

[14] E. Horowitz. Algorithms for partial fraction decomposition and rational function integration. *Proc. of SYMSAM'71, ACM Press*, pages 441–457, 1971.

[15] J. Hubbard and V. Moll. A geometric view of rational Landen transformation. *Bull. London Math. Soc.*, 35:293–301, 2003.

[16] D. Lazard and R. Rioboo. Integartion of rational functions: rational computation of the logarithmic part. *Journal of Symbolic Computation*, 9:113–116, 1990.

[17] D. Manna and V. Moll. A simple example of a new class of Landen transformations. *American Mathematical Montly*, 2006.

[18] H. McKean and V. Moll. *Elliptic Curves: Function Theory, Geometry, Arithmetic*. Cambridge University Press, New York, 1997.

[19] M. W. Ostrogradsky. De l'integration des fractions rationelles. *Bulletin de la Classe Physico-Mathematiques de l'Academie Imperiale des Sciences de St. Petersbourgh*, IV:145–167, 286–300, 1845.

[20] J. F. Ritt. Permutable rational functions. *Trans. Amer. Math. Soc.*, 25:399–448, 1923.

[21] M. Rothstein. A new algorithm for the integration of exponential and logarithmic functions. *Proc. of the* 1977 *MACSYMA Users Conference, NASA Pub., CP-2012*, pages 263–274, 1977.

[22] B. M. Trager. Algebraic factoring and rational function integration. *Proc. SYMSAC*, pages 219–226, 1976.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
*E-mail address*: `dmanna@tulane.edu`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118
*E-mail address*: `vhm@math.tulane.edu`