MODIFIED NÖRLUND POLYNOMIALS

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ABSTRACT. The modified Bernoulli numbers B_n^* considered by Zagier are generalized to modified Nörlund polynomials $B_n^{(\ell)*}$. For $\ell \in \mathbb{N}$, an explicit expression for the generating function for these polynomials is obtained. Evaluations of some spectacular integrals involving Chebyshev polynomials, and of a finite sum involving integrals of the Hurwitz zeta function are also obtained. New results about the ℓ-fold convolution of the square hyperbolic secant distribution are obtained, such as a differential-difference equation satisfied by a logarithmic moment and a closed-form expression in terms of the Barnes zeta function.

1. INTRODUCTION

The Bernoulli numbers, defined by the generating function

$$(1.1) \qquad \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1},$$

were extended by N. E. Nörlund [12, Ch. 6] to

$$(1.2) \qquad \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{z^n}{n!} = \left(\frac{z}{e^z - 1} \right)^{\alpha}.$$

Here $\alpha \in \mathbb{C}$. The coefficients $B_n^{(\alpha)}$ are called the *Nörlund polynomials* (these are indeed polynomials in α). The list $\{B_n^{(\alpha)} : n \geq 0\}$ begins with

$$(1.3) \qquad \left\{ 1, -\frac{\alpha}{2}, \frac{1}{12}\alpha(3\alpha - 1), -\frac{1}{8}\alpha^2(\alpha - 1), \frac{1}{240}\alpha(15\alpha^3 - 30\alpha^2 + 5\alpha + 2) \right\}.$$

For $\alpha \in \mathbb{N}$, the Nörlund polynomials are expressed as the α-fold convolutions of Bernoulli numbers. This follows from the recurrence

$$(1.4) \qquad B_n^{(\alpha)} = \sum_{j=0}^{n} \binom{n}{j} B_j^{(\alpha-1)} B_{n-j}, \quad \text{for } \alpha \geq 2,$$

obtained from (1.2), and the initial condition $B_n^{(1)} = B_n$.

Zagier [16] introduced a modification of the Bernoulli numbers via

$$(1.5) \qquad B_n^* = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r}, \quad n \in \mathbb{N},$$

and their polynomial version

$$(1.6) \qquad B_n^*(x) = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r(x)}{n+r},$$

was studied in detail in [7]. Here $B_n(x)$ is the Bernoulli polynomial with the generating function

$$(1.7) \qquad \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1},$$

and so along with (1.1), we have $B_n = B_n(0)$.

In particular, [7] establishes the formula

$$(1.8) \qquad \sum_{n=1}^{\infty} B_n^*(x) z^n = -\frac{1}{2} \log z - \frac{1}{2} \psi(z + 1/z - 1 - x)$$

for the generating function of the Zagier polynomials $B_n^*(x)$, viewed as a formal power series. Here

$$(1.9) \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function. The special case $x = 0$ yields

$$(1.10) \qquad \sum_{n=1}^{\infty} B_n^* z^n = -\frac{1}{2} \log z - \frac{1}{2} \psi(z + 1/z - 1).$$

In the present work, the Nörlund polynomials are modified in a similar way as Zagier's. These *modified Nörlund polynomials* are defined here by

$$(1.11) \qquad B_n^{(\alpha)*} := \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r^{(\alpha)}}{n+r}, \quad n \in \mathbb{N}.$$

The Zagier modification of the Bernoulli numbers (1.5) is the special case $\alpha = 1$. For $\alpha \in \mathbb{N}$, the main result of this paper is an expression for the generating function

$$(1.12) \qquad F_{B^*}(z; \alpha) = \sum_{n=1}^{\infty} B_n^{(\alpha)*} z^n,$$

involving derivatives of the digamma function as given in Theorem 1.2. This is a generalization of (1.10).

Notation. Standard notation is used throughout the paper.

1) The generalized binomial coefficients are defined by

$$\binom{x}{n} = \frac{1}{n!} x(x-1) \cdots (x-n+1),$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

2) The harmonic numbers are defined by

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

3) The gamma function is defined by the integral representation

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt$$

for $\operatorname{Re} z > 0$ and extended by analytic continuation. It satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$.

4) The digamma function $\psi(z)$ is defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z).$$

It satisfies

$$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

5) The Chebyshev polynomials of the first and second kind are defined, respectively, by their Binet representations [15]

$$T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right] \tag{1.13}$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[(x + \sqrt{x^2-1})^n - (x - \sqrt{x^2-1})^n\right]. \tag{1.14}$$

The work presented here is based on the symbolic notation

$$g(x+B) = \frac{\pi}{2}\int_{-\infty}^{\infty} g\left(x - \tfrac{1}{2} + iv\right) \operatorname{sech}^2(\pi v)\, dv. \tag{1.15}$$

The formula (1.15) is based on the fact that, if L_B is a random variable with the square secant hyperbolic distribution

$$\rho(x) = \frac{\pi}{2}\operatorname{sech}^2(\pi x), \tag{1.16}$$

then

$$B_n = \mathbb{E}\left(\imath L_B - \tfrac{1}{2}\right)^n = \int_{-\infty}^{\infty} \rho(u)\left(\imath u - \tfrac{1}{2}\right)^n\, du \tag{1.17}$$

so that, symbolically, with $g(x) = x^n$,

$$B_n = g(B). \tag{1.18}$$

This extends to Bernoulli polynomials as

$$B_n(x) = (B + x)^n = \mathbb{E}\left(x + \imath L_B - \tfrac{1}{2}\right)^n \tag{1.19}$$

and to any analytic function g as

$$\mathbb{E}\left[g(x - \tfrac{1}{2} + iL_B)\right] = \frac{\pi}{2}\int_{-\infty}^{\infty} g\left(x - \tfrac{1}{2} + iv\right) \operatorname{sech}^2(\pi v)\, dv. \tag{1.20}$$

This is complemented with the notation

$$f(x + U) = \int_0^1 f(x + u)\, du, \tag{1.21}$$

that corresponds to the average over a uniform distribution U on $[0, 1]$.

The symbolic form (1.15) is a restatement of the umbral approach described in [7]. The classical umbral calculus begins with a sequence $\{a_n\}$ and formally transforms it into powers $\mathfrak{a}^n$ of a new variable $\mathfrak{a}$, named the umbra of $\{a_n\}$. The

original sequence is then recovered by the evaluation map $\text{eval}\{\mathfrak{a}^n\} = a_n$. The *Bernoulli umbra*, studied in [7], is defined by the generating function

$$(1.22) \qquad \text{eval}\{\exp\left(t\mathfrak{B}(x)\right)\} = \frac{te^{xt}}{e^t - 1}$$

and it satisfies, with $\mathfrak{B} = \mathfrak{B}(0)$,

$$(1.23) \qquad -\mathfrak{B} = \mathfrak{B} + 1, \text{ and } (-\mathfrak{B})^n = \mathfrak{B}^n \text{ for } n \neq 1,$$

and

$$(1.24) \qquad \text{eval}\{\mathfrak{B}(x)\} = \text{eval}\{x + \mathfrak{B}\}.$$

For more properties of Bernoulli umbrae, the reader is referred to Gessel [10]. Theorem 2.3 in [7] states that the Bernoulli umbra coincides with a random variable $iL_B - \frac{1}{2}$, in the sense that,

$$(1.25) \qquad \text{eval}\{g(\mathfrak{B} + x)\} = \mathbb{E}\left[g(x - \tfrac{1}{2} + iL_B)\right],$$

for all admissible functions g.

Thus from (1.15), (1.20) and (1.25), one obtains the three equivalent notations

$$(1.26) \qquad g(x + B) = \mathbb{E}\left[g(x - \tfrac{1}{2} + iL_B)\right] = \text{eval}\{g(\mathfrak{B}(x))\},$$

and for brevity, we will mostly use the symbolic form $g(x + B)$.

Now take ℓ independent copies $\{L_{B_1}, \cdots, L_{B_\ell}\}$ of the random variable L_B. The density $\rho_\ell(x)$ associated to $L = L_{B_1} + \cdots L_{B_\ell}$ is then the ℓ-fold convolution of the density $\rho(x)$ of each summand. This is computed by the recurrence

$$(1.27) \qquad \rho_\ell(x) = \int_{-\infty}^{\infty} \rho_{\ell-1}(u)\rho_1(x - u)\,du,$$

starting with $\rho_1(x) = \rho(x)$. A direct computation of the densities $\{\rho_\ell\}$ is remarkably difficult. The case $\ell = 2$ is presented in Section 4. The case $\ell = 1$ and $g(x) = \log x$ of the formula

$$(1.28) \qquad \mathbb{E}[g(x - \tfrac{\ell}{2} + iL_{B_1} + iL_{B_2} + \cdots + iL_{B_\ell})] = \int_{-\infty}^{\infty} \rho_\ell(u)g(x - \tfrac{\ell}{2} + iu)\,du$$

was used in [7] to evaluate the generating functions of the modified Bernoulli numbers and of Zagier polynomials. In the umbral notation, this quantity can be written as

$$(1.29) \qquad \text{eval}\left[g(x + \mathfrak{B}_1 + \cdots + \mathfrak{B}_\ell)\right] = \text{eval}\left[g\left(\mathfrak{B}^{(\ell)}(x)\right)\right],$$

so that the umbra associated with the modified Nörlund polynomials is

$$(1.30) \qquad \mathfrak{B}^{(\ell)} = \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots + \mathfrak{B}_\ell.$$

This is extended here to compute the corresponding generating function of the modified Nörlund polynomials. A crucial step in the argument uses the following result which evaluates the logarithm of the umbra $\mathfrak{B}^{(\ell)}$.

Theorem 1.1. *Let $\ell \in \mathbb{N}$ be fixed. For $x \in \mathbb{R}$,*

$$\text{eval}\{\log \mathfrak{B}^{(\ell)}(x)\} = -H_{\ell-1} + \frac{d^{\ell-1}}{dx^{\ell-1}}\left\{\binom{x-1}{\ell-1}\psi\left(x - \left\lfloor\frac{\ell}{2}\right\rfloor\right)\right\}.$$

Here H_r is the harmonic number and $\psi(x)$ is the digamma function and $\left\lfloor\frac{\ell}{2}\right\rfloor$ denotes the floor function.

The following generating function for the modified Nörlund polynomials is now obtained from the previous theorem.

Theorem 1.2. *Let $\ell \in \mathbb{N}$ be fixed. The generating function*

$$F_{B^*}(z; \ell) = \sum_{n=1}^{\infty} B_n^{(\ell)*} z^n$$

for the modified Nörlund polynomials $B_n^{(\ell)*}$ is given by

$$F_{B^*}(z; \ell) = -\frac{1}{2} \log z + \frac{1}{2} \left[H_{\ell-1} - \frac{d^{\ell-1}}{dx^{\ell-1}} \left\{ \binom{x-1}{\ell-1} \psi\left(x - \left\lfloor \frac{\ell}{2} \right\rfloor\right) \right\} \right]$$

evaluated at $x = z + 1/z + \ell - 2$.

An alternate representation for eval$\{\log \mathfrak{B}^{(\ell)}(x)\}$ gives the following remarkable integral evaluation involving the density $\rho_\ell(x)$.

Theorem 1.3. *Let $\ell \in \mathbb{N}$ be fixed. Then*

$$\int_0^\infty \log(1 + bu^2)\, \rho_\ell(u)\, du =$$

$$- \log\left|x - \frac{\ell}{2}\right| - H_{\ell-1} + \frac{d^{\ell-1}}{dx^{\ell-1}} \left\{ \binom{x-1}{\ell-1} \psi\left(x - \left\lfloor \frac{\ell}{2} \right\rfloor\right) \right\}$$

with $b = (x - \ell/2)^{-2}$.

Theorems 1.2 and 1.3 readily give the following result. We record it as a theorem only to emphasize the link between the generating function of the modified Nörlund polynomials and the definite integral containing the density $\rho_\ell(x)$.

Theorem 1.4. *Let $\ell \in \mathbb{N}$ be fixed. The generating function*

$$F_{B^*}(z; \ell) = \sum_{n=1}^{\infty} B_n^{(\ell)*} z^n$$

for the modified Nörlund polynomials $B_n^{(\ell)*}$ is given by

$$F_{B^*}(z; \ell) = -\frac{1}{2} \log z - \frac{1}{2} \log\left|x - \frac{\ell}{2}\right| - \frac{1}{2} \int_0^\infty \log(1 + bu^2)\rho_\ell(u)\, du$$

with $b = (x - \ell/2)^{-2}$ and $x = z + 1/z + \ell - 2$.

Section 2 describes a symbolic formalism based on two probability densities. This is used in Section 3 to obtain an expression for the generating function of the Nörlund polynomials. Section 4 presents a family of densities that provide an alternative form of this generating function. These densities satisfy a differential-difference equation and the initial conditions are evaluated in Section 5. The last two sections uses the densities described above to evaluate some definite integrals involving Chebyshev polynomials and the Hurwitz zeta function. A direct evaluation of these examples seems out of the range of the current techniques of integration.

2. Some symbolic formalism

The definition of the digamma function as $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ immediately gives the evaluation

$$(2.1) \qquad \int_0^1 \psi(x+t)\,dt = \log x.$$

The inversion formula

$$(2.2) \qquad \psi(x) = \frac{\pi}{2} \int_{-\infty}^{\infty} \log(x - \tfrac{1}{2} + iu)\operatorname{sech}^2 \pi u\,du$$

was established in Theorem 2.5 of [7].

In the notation (1.21) and (1.15), (2.1) and (2.2) expresses the equivalence of the relations

$$(2.3) \qquad \psi(x+U) = \log x \text{ and } \psi(x) = \log(x+B).$$

This is now shown to be a particular case of a more general inversion formula.

Definition 2.1. For real-valued functions f and g, define in recursive form

$$(2.4) \qquad f(x + U^{(\ell)}) = f(x + U + U^{(\ell-1)}) \quad \text{for } \ell \geq 2,$$

with $U^{(1)} = U$, and similarly

$$(2.5) \qquad g(x + B^{(\ell)}) = g(x + B + B^{(\ell-1)}) \quad \text{for } \ell \geq 2,$$

with $B^{(1)} = B$.

In the lemma given below, this symbolic formalism is connected to anti-derivatives $F^{(\ell)}$ of the function f, defined as any function $F^{(\ell)}$ such that $\dfrac{d^\ell}{dx^\ell} F^{(\ell)}(x) = f(x)$, via the classical forward difference operator Δ defined by

$$(2.6) \qquad \Delta f(x) = f(x+1) - f(x).$$

It is clear that if f is a polynomial of degree ℓ, then $\Delta f(x)$ is also a polynomial and its degree is $\ell - 1$.

Lemma 2.2. *Let $F^{(\ell)}$ be an antiderivative of f of order ℓ. Then*

$$(2.7) \qquad f(x + U^{(\ell)}) = \Delta^\ell F^{(\ell)}(x).$$

Proof. The case $\ell = 1$ is straightforward since

$$(2.8) \qquad f(x+U) = \int_0^1 f(x+u)\,du = F(x+1) - F(x)$$

by the Fundamental Theorem of Calculus. The inductive step is

$$\begin{aligned}
f\left(x + U^{(\ell+1)}\right) &= f\left(x + U^{(\ell)} + U\right) \\
&= \Delta^\ell F^{(\ell)}(x+U) \\
&= \Delta^\ell \left[F^{(\ell+1)}(x+1) - F^{(\ell+1)}(x) \right] \\
&= \Delta^{\ell+1} F^{(\ell+1)}(x).
\end{aligned}$$

$\square$

The next result is a generalization of (2.3): it shows that the symbols U and B invert each other. The proof uses the evaluation of the definite integral

$$(2.9) \qquad \int_0^\infty \frac{\cos zv}{\cosh^2 \pi v}\, dv = \frac{z}{2\pi}\left(\sinh\frac{z}{2}\right)^{-1},$$

which is obtained from entry 3.982.2 in [11]:

$$(2.10) \qquad \int_0^\infty \frac{\cos ax}{\cosh^2 \beta x}\, dx = \frac{\pi a}{2\beta^2}\left(\sinh\frac{\pi a}{2\beta}\right)^{-1}, \qquad \operatorname{Re}\beta > 0,\ a > 0.$$

A proof of this entry will appear in [6].

Theorem 2.3. *For any admissible formal power series,*

$$(2.11) \qquad g(x) = f(x + U) \text{ is equivalent to } f(x) = g(x + B).$$

Proof. In view of linearity, it suffices to consider the case $f(x) = x^n$. Start with the generating function

$$\sum_{n=0}^\infty \frac{(x+U)^n}{n!} z^n = e^{z(x+U)} = \int_0^1 e^{z(x+u)}du = e^{zx}\frac{e^z - 1}{z},$$

and

$$\begin{aligned}
\sum_{n=0}^\infty \frac{(x+B)^n}{n!} z^n &= e^{z(x+B)} = \frac{\pi}{2}\int_{-\infty}^\infty e^{z\left(x+iv-\frac{1}{2}\right)}\operatorname{sech}^2(\pi v)\, dv \\
&= \frac{\pi}{2}e^{z\left(x-\frac{1}{2}\right)}\int_{-\infty}^\infty e^{izv}\operatorname{sech}^2(\pi v)\, dv \\
&= \pi e^{z\left(x-\frac{1}{2}\right)}\int_0^\infty \frac{\cos(zv)}{\cosh^2(\pi v)}\, dv.
\end{aligned}$$

The evaluation (2.10) gives

$$(2.12) \qquad \sum_{n=0}^\infty \frac{(x+B)^n}{n!} z^n = e^{zx}\frac{z}{e^z - 1}.$$

Now assume first that $g(x) = f(x + U)$, i.e., $g(x) = (x+U)^n$. Then

$$\begin{aligned}
\sum_{n=0}^\infty \frac{g(x+B)z^n}{n!} &= \sum_{n=0}^\infty \frac{(x+B+U)^n}{n!}z^n \\
&= e^{z(x+B)}\frac{e^z - 1}{z} \\
&= e^{zx}\frac{z}{e^z - 1}\frac{e^z - 1}{z} = e^{zx}.
\end{aligned}$$

From here it follows that $(x + B + U)^n = x^n$. The other direction is established in a similar form. $\square$

Note 2.1. A direct extension gives the equivalence of the statements

$$(2.13) \qquad g(x) = f(x + U^{(\ell)}) \text{ and } f(x) = g(x + B^{(\ell)}), \qquad \text{for all } \ell \in \mathbb{N},$$

which can be proved by induction.

3. The generating function of the modified Nörlund polynomials

This section uses the results of the previous section to prove an expression for the horizontal generating function of the modified Nörlund polynomials $B_n^{(\ell)*}$ as a formal power series. Here ℓ is a fixed positive integer. This generating function is defined by

$$(3.1) \qquad F_{B^*}(z; \ell) = \sum_{n=1}^{\infty} B_n^{(\ell)*} z^n.$$

Lemma 3.1. *Let $\psi(x)$ be the digamma function and H_ℓ the ℓ-th harmonic number. Then for $\ell \geq 1$ and $-1 \leq p \leq \ell - 1$,*

$$(3.2) \qquad \Delta^\ell \left[\binom{x+p}{\ell} \psi(x) \right] = H_\ell + \psi(x+p+1).$$

Proof. The result is established first for $p = 0$. Define

$$(3.3) \qquad h_\ell(x) = \Delta^\ell \left[\binom{x}{\ell} \psi(x) \right]$$

and observe that

$$
\begin{aligned}
h_{\ell+1}(x) - h_\ell(x) &= \Delta^{\ell+1} \left[\binom{x}{\ell+1} \psi(x) \right] - \Delta^\ell \left[\binom{x}{\ell} \psi(x) \right] \\
&= \Delta^\ell \Delta \left[\binom{x}{\ell+1} \psi(x) \right] - \Delta^\ell \left[\binom{x}{\ell} \psi(x) \right] \\
&= \Delta^\ell \left[\binom{x+1}{\ell+1} \psi(x+1) - \binom{x}{\ell+1} \psi(x) - \binom{x}{\ell} \psi(x) \right].
\end{aligned}
$$

The identity

$$(3.4) \qquad \binom{x+1}{\ell+1} = \binom{x}{\ell+1} + \binom{x}{\ell}$$

gives

$$
\begin{aligned}
h_{\ell+1}(x) - h_\ell(x) &= \Delta^\ell \left[\binom{x}{\ell+1} (\psi(x+1) - \psi(x)) + \binom{x}{\ell} (\psi(x+1) - \psi(x)) \right] \\
&= \Delta^\ell \left[\frac{1}{x} \left(\binom{x}{\ell+1} + \binom{x}{\ell} \right) \right] \\
&= \Delta^\ell \left[\frac{(x-1) \cdots (x-\ell)}{(\ell+1)!} + \frac{(x-1) \cdots (x-\ell+1)}{\ell!} \right]
\end{aligned}
$$

The second fraction is a polynomial in x of degree $\ell - 1$. Therefore Δ^ℓ annihilates it. The first fraction is a polynomial of degree ℓ and only its leading term survives the application of Δ^ℓ. This leads to the difference equation

$$(3.5) \qquad h_{\ell+1}(x) - h_\ell(x) = \Delta^\ell \frac{x^\ell}{(\ell+1)!} = \frac{1}{\ell+1},$$

since $\Delta^\ell x^\ell = \ell!$. The latter follows directly from Lemma 2.2: indeed, choosing $f(x) = 1$ produces $F^{(\ell)}(x) = x^\ell / \ell!$ and therefore

$$(3.6) \qquad \Delta^\ell \frac{x^\ell}{\ell!} = f(x + U^{(\ell-1)}) = 1,$$

which gives the result. Now write (3.5) as

$$(3.7) \qquad h_{\ell+1}(x) - \psi(\ell+2) = h_\ell(x) - \psi(\ell+1),$$

so that

$$(3.8) \qquad h_\ell(x) = h_1(x) + \psi(\ell+1) - \psi(2).$$

Now

$$(3.9) \qquad h_1(x) = \Delta\left[\binom{x}{1}\psi(x)\right] = 1 + \psi(x+1)$$

gives the stated result for $p = 0$.

Now assume $p \neq 0$ and that $1 \leq p \leq \ell - 1$. Observe that

$$\Delta^\ell\left[\binom{x+p}{\ell}(\psi(x+p) - \psi(x))\right] = \Delta^\ell\left[\binom{x+p}{\ell}\left(\frac{1}{x+p-1} + \cdots + \frac{1}{x}\right)\right]$$

$$= \frac{1}{\ell!}\Delta^\ell\left[\prod_{u=0}^{\ell-1}(x+p-u) \times \left(\frac{1}{x+p-1} + \cdots + \frac{1}{x}\right)\right]$$

The bounds $1 \leq p \leq \ell - 1$ show that the last expression is actually a polynomial of degree $\ell - 1$. One can also easily check that when $p = -1$, $\binom{x+p}{\ell}(\psi(x+p) - \psi(x))$ is a polynomial of degree $\ell - 1$. This implies that for $-1 \leq p \leq \ell - 1, p \neq 0$,

$$(3.10) \qquad \Delta^\ell\left[\binom{x+p}{\ell}(\psi(x+p) - \psi(x))\right] = 0.$$

It follows that

$$\Delta^\ell\left[\binom{x+p}{\ell}\psi(x)\right] = \Delta^\ell\left[\binom{x+p}{\ell}\psi(x+p)\right]$$

$$= h_\ell(x+p)$$

$$= H_\ell + \psi(x+p+1),$$

as can be seen from (3.8) and (3.9). This completes the argument. $\qquad \square$

The proof of Theorem 1.1 is given next.

Proof. Using the symbolic operator B, the left-hand side of Theorem 1.1 can be written as $\log(x + B^{(\ell)})$. Let $f(x)$ denote the right-hand side of Theorem 1.1, i.e.,

$$(3.11) \qquad f(x) = -H_{\ell-1} + \frac{d^{\ell-1}}{dx^{\ell-1}}\left[\binom{x-1}{\ell-1}\psi\left(x - \left\lfloor\frac{\ell}{2}\right\rfloor\right)\right].$$

Using (2.13), it suffices to prove

$$f\left(x + U^{(\ell-1)}\right) = \log(x + B).$$

However from (2.3),

$$(3.12) \qquad \log(x + B) = \psi(x).$$

So we only need to show that

$$f\left(x + U^{(\ell-1)}\right) = \psi(x).$$

Now Lemma 2.2 gives

$$(3.13) \qquad f(x + U^{(\ell-1)}) = \Delta^{\ell-1}F^{(\ell-1)}(x),$$

and writing $f(x)$ as

$$(3.14) \qquad f(x) = \frac{d^{\ell-1}}{dx^{\ell-1}} \left[\binom{x-1}{\ell-1} \psi \left(x - \left\lfloor \frac{\ell}{2} \right\rfloor \right) - \frac{x^{\ell-1}}{(\ell-1)!} H_{\ell-1} \right]$$

produces

$$(3.15) \qquad F^{(\ell-1)}(x) = \binom{x-1}{\ell-1} \psi \left(x - \left\lfloor \frac{\ell}{2} \right\rfloor \right) - \frac{x^{\ell-1}}{(\ell-1)!} H_{\ell-1}.$$

Then

$$(3.16) \qquad \Delta^{\ell-1} F^{(\ell-1)}(x) = -H_{\ell-1} + \Delta^{\ell-1} \left[\binom{x-1}{\ell-1} \psi \left(x - \left\lfloor \frac{\ell}{2} \right\rfloor \right) \right]$$

and (3.13) gives

$$(3.17) \qquad f(x + U^{(\ell-1)}) = -H_{\ell-1} + \Delta^{\ell-1} \left[\binom{x-1}{\ell-1} \psi \left(x - \left\lfloor \frac{\ell}{2} \right\rfloor \right) \right].$$

Now Lemma 3.1, with ℓ replaced by $\ell - 1$, x replaced by $x - \left\lfloor \frac{\ell}{2} \right\rfloor$ and $p = \left\lfloor \frac{\ell}{2} \right\rfloor - 1$, yields

$$(3.18) \qquad f(x + U^{(\ell-1)}) = \psi(x).$$

This completes the proof. □

The proof of Theorem 1.2, which expresses the generating function for the modified Nörlund polynomials, is now given.

Proof. The proof of the identity

$$(3.19) \qquad F_{B^*}(z) = \sum_{n=1}^{\infty} B_n^* z^n = -\text{eval} \left\{ \tfrac{1}{2} \log \left((1-z)^2 - z\mathfrak{B} \right) \right\}$$

given in [7, Equation (3.4)] can be adapted to derive, in a similar manner, the relation

$$(3.20) \qquad F_{B^*}(z; \ell) = \sum_{n=1}^{\infty} B_n^{(\ell)*} z^n = -\text{eval} \left\{ \frac{1}{2} \log \left((1-z)^2 - z\mathfrak{B}^{(\ell)} \right) \right\}.$$

This is described next. Basic facts of umbral calculus, namely $-\mathfrak{B}^{(\ell)} = \mathfrak{B}^{(\ell)} + \ell$ and $x + \mathfrak{B}^{(\ell)} = \mathfrak{B}^{(\ell)}(x)$, give

$$(3.21) \quad \sum_{n=1}^{\infty} B_n^{(\ell)*} z^n \quad = \quad -\frac{1}{2} \log z - \frac{1}{2} \text{eval} \left\{ \log \left(z + \frac{1}{z} - 2 - \mathfrak{B}^{(\ell)} \right) \right\}$$

$$= \quad -\frac{1}{2} \log z - \frac{1}{2} \text{eval} \left\{ \log \left(z + \frac{1}{z} - 2 + \mathfrak{B}^{(\ell)} + \ell \right) \right\}$$

$$= \quad -\frac{1}{2} \log z - \frac{1}{2} \text{eval} \left\{ \log \mathfrak{B}^{(\ell)} \left(z + \frac{1}{z} + \ell - 2 \right) \right\}.$$

The final step uses Theorem 1.1. □

The proof of Theorem 1.3 is presented next.

Proof. Start with

$$
\begin{aligned}
\text{eval}\{\log \mathfrak{B}^{(\ell)}(x)\} &= \text{eval}\{\log (x + \mathfrak{B}_1 + \cdots + \mathfrak{B}_\ell)\} \\
&= \mathbb{E}\left[\log \left(x - \tfrac{\ell}{2} + i(L_{B_1} + \cdots + L_{B_\ell})\right)\right].
\end{aligned}
$$

Introduce the notation $L = L_{B_1} + \cdots + L_{B_\ell}$ and since the density ρ_ℓ is an even function, L and $-L$ have the same distribution. Therefore, with $b = (x - \ell/2)^{-2}$,

$$
\begin{aligned}
\text{eval}\{\log \mathfrak{B}^{(\ell)}(x)\} &= \frac{1}{2}\mathbb{E}\left[\log\left(\left(x - \frac{\ell}{2}\right)^2 + L^2\right)\right] \\
&= \log\left|x - \frac{\ell}{2}\right| + \frac{1}{2}\mathbb{E}\left[\log(1 + bL^2)\right] \\
&= \log\left|x - \frac{\ell}{2}\right| + \frac{1}{2}\int_{-\infty}^{\infty} \log(1 + bu^2)\rho_\ell(u)\, du
\end{aligned}
$$

$$
(3.22) \qquad\qquad = \log\left|x - \frac{\ell}{2}\right| + \int_0^{\infty} \log(1 + bu^2)\rho_\ell(u)\, du,
$$

since ρ_ℓ is an even function of u. The result now follows from Theorem 1.1. $\qquad\square$

4. A FAMILY OF DENSITIES AND A DIFFERENTIAL-DIFFERENCE EQUATION

This section discusses the densities $\rho_n(x)$ defined by the recurrence

$$
(4.1) \qquad\qquad \rho_n(x) = \int_{-\infty}^{\infty} \rho_{n-1}(y)\rho_1(x - y)dy
$$

with initial condition

$$
(4.2) \qquad\qquad \rho_1(x) = \rho(x) = \frac{\pi}{2}\text{sech}^2(\pi x).
$$

These densities provide the evaluation

$$
(4.3) \qquad \mathbb{E}\left[g\left(x - \tfrac{\ell}{2} + i\sum_{j=1}^{\ell} B_j\right)\right] = \int_{-\infty}^{\infty} g(x - \tfrac{1}{2} + iv)\rho_\ell(v)dv.
$$

In particular, the generating function of the Nörlund polynomials is linked to these densities via Theorem 1.4. Some properties of these densities are described next.

Lemma 4.1. *The Fourier transform of $\rho_1(x)$ is given by*

$$
(4.4) \qquad\qquad \widehat{\rho_1}(\xi) = \frac{\pi\xi}{\sinh \pi\xi}.
$$

Proof. The Fourier transform is given by

$$
\begin{aligned}
\widehat{\rho_1}(\xi) &= \int_{-\infty}^{\infty} \frac{\pi}{2}\text{sech}^2(\pi x)e^{-2\pi i x\xi}\, dx \\
&= \pi \int_0^{\infty} \frac{\cos(2\pi x\xi)}{\cosh^2(\pi x)}\, dx,
\end{aligned}
$$

and the result follows by using (2.9). $\qquad\square$

Corollary 4.1. *The Fourier transform of $\rho_\ell(x)$ is given by*

$$
(4.5) \qquad\qquad \widehat{\rho_\ell}(\xi) = \left(\frac{\pi\xi}{\sinh \pi\xi}\right)^{\ell}.
$$

Proof. This follows directly from the fact that Fourier transform converts convolutions into products. $\square$

The Fourier inversion formula now gives a representation for the density $\rho_\ell(x)$ as

$$(4.6) \qquad \rho_\ell(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{y}{\sinh y} \right)^\ell e^{2ixy} \, dy.$$

Note 4.2. J. Pitman and M. Yor [13, p. 299] studied the function

$$(4.7) \qquad \phi_\ell(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{y}{\sinh y} \right)^\ell e^{ixy} dy$$

as part of their study on infinitely divisible distributions generated by Lévy processes associated with hyperbolic functions. The expression (4.6) shows that

$$(4.8) \qquad \rho_\ell(x) = 2\phi_\ell(2x).$$

These authors show that ϕ_t satisfies the differential-difference equation

$$(4.9) \quad \ell(\ell+1)\phi_{\ell+2}(x) = (x^2 + \ell^2)\phi_\ell''(x) + 2(\ell+2)x\phi_\ell'(x) + (\ell+1)(\ell+2)\phi_\ell(x).$$

Note 4.3. The authors of [13] also consider the transform

$$(4.10) \qquad \psi_\ell(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\cosh y} \right)^\ell e^{ixy} dy$$

and prove the explicit formulae

$$(4.11) \qquad \psi_\ell(x) = \frac{2^{\ell-2}}{\pi\Gamma(\ell)} \left| \Gamma\left(\frac{\ell + ix}{2} \right) \right|^2.$$

Then, they state '*we do not know of any explicit formula for ϕ_ℓ like (4.11) valid for general $\ell > 0$*'.

Note 4.4. The density functions $\rho_\ell(x)$ have also appeared in Airault [1, p. 2109, (1.52), (1.53)]. This author proves that

$$(4.12) \qquad \rho_{2\ell}(x) = \frac{\pi^{1-2\ell}}{2(2\ell-1)!} \frac{d^{2\ell}}{dx^{2\ell}} \left[\frac{Q_{2\ell-1}(\pi x)}{\tanh(\pi x)} \right]$$

$$\rho_{2\ell+1}(x) = \frac{\pi^{-2\ell}}{2(2\ell)!} \frac{d^{2\ell+1}}{dx^{2\ell+1}} \left[Q_{2\ell}(\pi x) \tanh(\pi x) \right]$$

where

$$(4.13) \qquad Q_{2\ell}(x) = \prod_{\substack{1 \le j \le 2\ell-1 \\ j \text{ odd}}} \left(x^2 + \frac{\pi^2 j^2}{4} \right)$$

and

$$(4.14) \qquad Q_{2\ell+1}(x) = x \prod_{1 \le j \le \ell} (x^2 + j^2\pi^2).$$

The differential-difference equation (4.9) produces

$$(4.15) \quad \ell(\ell+1)\rho_{\ell+2}(x) = \frac{(4x^2 + \ell^2)}{4}\rho_\ell''(x) + 2x(\ell+2)\rho_\ell'(x) + (\ell+1)(\ell+2)\rho_\ell(x),$$

so that $\rho_\ell(x)$ can be obtained from (4.15) and the initial conditions $\rho_1(x)$ in (4.2) and $\rho_2(x)$. Even though the expression for $\rho_2(x)$ is well-known [13, p. 312, Table 6], it is derived here for the sake of completeness.

Lemma 4.2. *The density function ρ_2 is given by*

$$\text{(4.16)} \qquad \rho_2(x) = \frac{\pi}{\sinh^2(\pi x)} \left(\pi x \, \coth(\pi x) - 1 \right).$$

Proof. The relation (1.27) gives

$$
\begin{aligned}
\rho_2(x) &= \int_{-\infty}^{\infty} \rho_1(u)\rho_1(x-u)\,du \\
&= \frac{\pi^2}{4} \int_{-\infty}^{\infty} \frac{du}{\cosh^2(\pi u)\,\cosh^2(\pi(x-u))} \\
&= \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{dt}{\cosh^2 t\,\cosh^2(\pi x - t)}.
\end{aligned}
$$

The change of variable $w = e^{2t}$ gives

$$\text{(4.17)} \qquad \rho_2(x) = 2\pi \int_0^{\infty} \frac{w\,dw}{(w+1)^2(\alpha + \beta w)^2},$$

with $\alpha = e^{\pi x}$ and $\beta = e^{-\pi x}$. The final integral is evaluated by partial fractions to produce the stated result. $\qquad\square$

The higher densities $\rho_\ell(x), \ell > 2$, can now be computed via (4.15) and the expressions in (4.2) and (4.16).

The next step is to show that the integral appearing in Theorem 1.3 satisfies a differential-difference equation.

Theorem 4.3. *The integral*

$$z_\ell(b) := \int_0^{\infty} \log(1 + bu^2)\, \rho_\ell(u)\,du$$

satisfies the differential-difference equation

$$\ell(\ell+1)y_{\ell+2}(x+1) = x(x-\ell)y_\ell''(x) + 2(\ell+1)\left(x - \frac{\ell}{2}\right)y_\ell'(x)$$

$$\text{(4.18)} \qquad\qquad\qquad + \ell(\ell+1)y_\ell(x) + \frac{\ell^2}{4\left(x - \frac{\ell}{2}\right)^2}$$

for $b = (x - \ell/2)^{-2}$.

Proof. Let $b > 0$. Start with (4.15), i.e.,

$$\ell(\ell+1)\rho_{\ell+2}(u) = \left(u^2 + \frac{\ell^2}{4}\right)\rho_\ell''(u) + 2u(\ell+2)\rho_\ell'(u) + (\ell+1)(\ell+2)\rho_\ell(u).$$

Multiply both sides by $\log(1 + bu^2)$ and integrate both sides from 0 to ∞ to obtain

$$\ell(\ell+1)z_{\ell+2}(b) = \int_0^{\infty}\left(u^2 + \frac{\ell^2}{4}\right)\log(1+bu^2)\,\rho_\ell''(u)\,du$$

$$\text{(4.19)} \qquad\qquad + 2(\ell+2)\int_0^{\infty} u\log(1+bu^2)\,\rho_\ell'(u)\,du + (\ell+1)(\ell+2)z_\ell(b).$$

Let

$$I_1(b,\ell) := \int_0^\infty u \log(1 + bu^2)\, \rho'_\ell(u)\, du,$$

(4.20) $$I_2(b,\ell) := \int_0^\infty \left(u^2 + \frac{\ell^2}{4}\right) \log(1 + bu^2)\, \rho''_\ell(u)\, du.$$

Consider $I_1(b,\ell)$ first. Integration by parts yields

(4.21)

$$I_1(b,\ell) = \left[u\log(1 + bu^2)\rho_\ell(u)\right]_0^\infty - \int_0^\infty \left(\frac{2bu^2}{1 + bu^2} + \log(1 + bu^2)\right)\rho_\ell(u)\, du.$$

Note that $\rho_\ell(t) \to 0$ as $t \to \infty$. This is easily seen for ρ_1 since

(4.22) $$\rho_1(t) = \frac{\pi}{2}\operatorname{sech}^2(\pi t) = \frac{2\pi e^{-2\pi t}}{(1 + e^{-2\pi t})^2} \to 0 \ \text{ as } \ t \to \infty.$$

For $\ell \geq 2$, use the definition of $\rho_\ell(t)$ in (1.27), and the above asymptotic for ρ_1, along with Lebesgue's dominated convergence theorem to deduce that $\rho_\ell(t) \to 0$ as $t \to \infty$. As $t \to 0$, it is easy to see that the densities $\rho_\ell(t)$ are finite.

This implies that the boundary terms in (4.21) vanish so that

$$I_1(b,\ell) = -z_\ell(b) - 2b \int_0^\infty \rho_\ell(u) \frac{d}{db} \log(1 + bu^2)\, du$$

(4.23) $$= -z_\ell(b) - 2b\frac{d}{db} z_\ell(b),$$

where differentiation (with respect to b) under the integral sign was employed in the last step.

Now consider $I_2(b,\ell)$, use integration by parts twice, and note that the boundary terms again vanish, thereby giving

$$I_2(b,\ell) = \int_0^\infty \left\{\frac{b(\ell^2 + (20 - b\ell^2)u^2 + 12bu^4)}{2(1 + bu^2)^2} + 2\log(1 + bu^2)\right\}\rho_\ell(u)\, du$$

(4.24) $$= 2z_\ell(b) + \frac{b}{2}\int_0^\infty \frac{(\ell^2 + (20 - b\ell^2)u^2 + 12bu^4)}{(1 + bu^2)^2}\rho_\ell(u)\, du.$$

Next, use the following representation

(4.25) $$\frac{(\ell^2 + (20 - b\ell^2)u^2 + 12bu^4)}{(1 + bu^2)^2} = \frac{\ell^2}{(1 + bu^2)^2} + \frac{(8 - b\ell^2)u^2}{(1 + bu^2)^2} + \frac{12u^2}{1 + bu^2}$$

to rewrite the above expression for $I_2(b,\ell)$ in the form

$$I_2(b,\ell) = 2z_\ell(b) + \frac{b}{2}\left\{\ell^2 \int_0^\infty \frac{\rho_\ell(u)}{(1 + bu^2)^2}\, du + (8 - b\ell^2)\int_0^\infty \frac{u^2\rho_\ell(u)}{(1 + bu^2)^2}\, du\right.$$

(4.26) $$\left. + 12\int_0^\infty \frac{u^2\rho_\ell(u)}{1 + bu^2}\, du\right\}.$$

As shown before,

(4.27) $$\int_0^\infty \frac{u^2\rho_\ell(u)}{1 + bu^2}\, du = \frac{d}{db} z_\ell(b).$$

Since

$$\frac{\rho_\ell(u)}{1 + bu^2} = \rho_\ell(u) - b\frac{u^2\rho_\ell(u)}{1 + bu^2},$$

and ρ_ℓ, being a probability density, satisfies $\int_{-\infty}^{\infty} \rho_\ell(u) = 1$, it is seen that

$$(4.28) \qquad \int_0^\infty \frac{\rho_\ell(u)}{1+bu^2} \, du = \frac{1}{2} - b \int_0^\infty \frac{u^2 \rho_\ell(u)}{1+bu^2} \, du = \frac{1}{2} - b \frac{d}{db} z_\ell(b).$$

Differentiation (with respect to b) under the integral sign then gives

$$(4.29) \qquad \int_0^\infty \frac{u^2 \rho_\ell(u)}{(1+bu^2)^2} \, du = b \frac{d^2}{db^2} z_\ell(b) + \frac{d}{db} z_\ell(b).$$

Similarly it can be shown that

$$(4.30) \qquad \int_0^\infty \frac{\rho_\ell(u)}{(1+bu^2)^2} \, du = \frac{1}{2} - 2b \frac{d}{db} z_\ell(b) - b^2 \frac{d^2}{db^2} z_\ell(b).$$

Now substitute (4.27), (4.29) and (4.30) in (4.26) and simplify to obtain

$$(4.31) \quad I_2(b,\ell) = b^2(4 - b\ell^2)\frac{d^2}{db^2} z_\ell(b) + b\left(10 - \frac{3b\ell^2}{2}\right)\frac{d}{db} z_\ell(b) + 2 z_\ell(b) + \frac{b\ell^2}{4}.$$

Then substitute (4.23) and (4.31) in (4.19) to deduce that

$$\ell(\ell+2) z_{\ell+2}(b) = b^2(4 - b\ell^2)\frac{d^2}{db^2} z_\ell(b) + 2b\left(1 - 2\ell - \frac{3b\ell^2}{4}\right)\frac{d}{db} z_\ell(b)$$

$$(4.32) \qquad\qquad\qquad + \ell(\ell+1) z_\ell(b) + \frac{b\ell^2}{4}.$$

Now let $b = (x - \ell/2)^{-2}$ as in Theorem 1.3, so that defining

$$y_\ell(x) := z_\ell(b),$$

and replacing ℓ by $\ell+2$, gives $z_{\ell+2}((x-1-\ell/2)^{-2}) = y_{\ell+2}(x)$, and hence

$$(4.33) \qquad\qquad z_{\ell+2}(b) = y_{\ell+2}(x+1).$$

A direct computation now gives

$$\frac{d}{db} z_\ell(b) = -\frac{1}{2}\left(x - \frac{\ell}{2}\right)^3 y_\ell'(x),$$

$$(4.34) \qquad \frac{d^2}{db^2} z_\ell(b) = \frac{1}{4}\left(x - \frac{\ell}{2}\right)^6 y_\ell''(x) + \frac{3}{4}\left(x - \frac{\ell}{2}\right)^5 y_\ell'(x),$$

where the prime denotes differentiation with respect to x. Finally, substitute (4.33) and (4.34) in (4.32) to arrive at (4.18). $\qquad\square$

Remark: The case $\ell = 1$ of Theorem 1.3 was derived in [7, Equation (2.28)] and an elementary proof of the case $\ell = 2$ is given in the next section. The differential-difference equation (4.18) then produces the values of

$$\int_0^\infty \log(1 + bu^2)\, \rho_\ell(u)\, du$$

for any $\ell > 2$.

5. The special case $\ell = 2$ of the generating function of the Nörlund polynomials

In this section, we present a different proof of Theorem 1.2 for $\ell = 2$ which was, in fact, the genesis of this project. It involves brute force verification of Theorem 1.3 when $\ell = 2$. It is then used along with the result in (3.22), namely,

$$(5.1) \qquad \operatorname{eval}\{\log \mathfrak{B}^{(2)}(x)\} = \log|x - 1| + \int_0^\infty \log(1 + bu^2)\rho_2(u)\, du$$

with $b = (x - 1)^{-2}$, and the special case of (3.21), namely,

$$(5.2) \qquad \sum_{n=1}^\infty B_n^{(2)*} z^n = -\frac{1}{2}\log z - \frac{1}{2}\operatorname{eval}\left\{\log \mathfrak{B}^{(2)}\left(z + \frac{1}{z}\right)\right\}.$$

The point to illustrate here is that these calculations soon become out of reach for large values of ℓ. In fact, the case $\ell = 3$ itself required six different integrals to be evaluated in order to arrive at Theorem 1.3 through the direct computation of the integral. At the end of the previous section, another way of calculating these integrals for all ℓ through a differential-difference equation was given. However, this being a recursive way, not only does it not give an explicit formula but also for higher values of ℓ, evaluating the integrals this way is a cumbersome process. These shortcomings are what led us to seek a new representation for $\operatorname{eval}\{\log \mathfrak{B}^{(\ell)}(x)\}$, namely Theorem 1.1, which gives an explicit formula for these integrals, avoiding messy calculations at the same time.

Proposition 5.1. *Let $a \neq 0$ and*

$$(5.3) \qquad I(a) = \int_0^\infty \frac{(x \coth x - 1)\log(1 + a^2 x^2)}{\sinh^2 x}\, dx.$$

Then

$$(5.4) \qquad I(a) = -\log c - 1 + \psi(c) + c\,\psi'(c),$$

with $c = \dfrac{1}{\pi a}$.

Proof. To evaluate this integral, observe first that

$$(5.5) \qquad \frac{d}{dx}\left(\coth x - \frac{x}{\sinh^2 x}\right) = \frac{2(x \coth x - 1)}{\sinh^2 x}$$

and write

$$(5.6) \qquad I(a) = \frac{1}{2}\int_0^\infty \log(1 + a^2 x^2)\frac{d}{dx}\left(\coth x - \frac{x}{\sinh^2 x}\right) dx.$$

In order to integrate by parts and guarantee the convergence of the boundary terms, write the integral as

$$(5.7) \qquad I(a) = \frac{1}{2}\int_0^\infty \log(1 + a^2 x^2)\frac{d}{dx}\left(\coth x - 1 - \frac{x}{\sinh^2 x}\right) dx.$$

Integrate by parts and verify that the boundary terms vanish to produce

$$(5.8) \qquad I(a) = a^2\left(I_1(a) + I_2(a)\right)$$

with

$$(5.9) \qquad I_1(a) = \int_0^\infty \frac{x(1 - \coth x)}{1 + a^2 x^2}\, dx \text{ and } I_2(a) = \int_0^\infty \frac{x^2\, dx}{(1 + a^2 x^2)\sinh^2 x}.$$

The evaluation of $I_1(a)$ is described first. Write it as

$$\begin{aligned}
I_1(a) &= -\int_0^\infty e^{-x} \frac{x}{\sinh x} \frac{dx}{1+a^2x^2} \\
&= -2\int_0^\infty \frac{x\,dx}{(1+a^2x^2)(e^{2x}-1)} \\
&= \frac{1}{a^2}\left[\psi\left(\frac{1}{\pi a}\right) + \frac{\pi a}{2} + \log(\pi a)\right]
\end{aligned}$$

using Entry 3.415.1 of [11]:

$$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}-1)} = \frac{1}{2}\left[\log\left(\frac{\beta\mu}{2\pi}\right) - \frac{\pi}{\beta\mu} - \psi\left(\frac{\beta\mu}{2\pi}\right)\right], \qquad \mathrm{Re}\,\beta > 0, \ \mathrm{Re}\,\mu > 0,$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function. A direct proof of this entry and some generalizations appear in [4].

To evaluate $I_2(a)$ write it as

$$\begin{aligned}
(5.10) \qquad I_2(a) &= 4\int_0^\infty \frac{x^2 e^{2x}\,dx}{(1+a^2x^2)(e^{2x}-1)^2} \\
&= -2\int_0^\infty \frac{x^2}{1+a^2x^2}\frac{d}{dx}\left(\frac{1}{e^{2x}-1}\right)dx.
\end{aligned}$$

Integration by parts and a simple scaling produces

$$(5.11) \qquad I_2(a) = \frac{4}{a^4\pi^2}\int_0^\infty \frac{x\,dx}{(x^2+c^2)^2(e^{2\pi x}-1)}$$

with $c = 1/(\pi a)$. Entry 3.415.2 in [11], established in [4], states that

$$(5.12) \qquad \int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2(e^{2\pi x}-1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta}\psi'(\beta),$$

which gives

$$(5.13) \qquad I_2(a) = -\frac{\pi}{2a} - \frac{1}{a^2} + \frac{1}{\pi a^3}\psi'\left(\frac{1}{\pi a}\right).$$

Replacing the values of $I_1(a)$ and $I_2(a)$ in (5.8) gives the result. $\qquad\square$

We now obtain Theorem 1.2 for $\ell = 2$ using the above proposition. To that end, let $x = \pi u$ and $a^2 = b/\pi^2$ in (5.3) and use Lemma 4.16 to find that

$$\int_0^\infty \rho_2(u)\log(1+bu^2)\,du = \psi\left(\frac{1}{\sqrt{b}}\right) + \frac{1}{\sqrt{b}}\psi'\left(\frac{1}{\sqrt{b}}\right) + \frac{1}{2}\log b - 1.$$

The above equation, along with (5.1) and the fact that $b = (x-1)^{-2}$, yields

$$\mathrm{eval}\left\{\log \mathfrak{B}^{(2)}(x)\right\} = \psi(|x-1|) + |x-1|\,\psi'(|x-1|) - 1.$$

Hence for $x \geq 1$, we have

$$(5.14) \qquad \mathrm{eval}\left\{\log \mathfrak{B}^{(2)}(x)\right\} = \psi(x-1) + (x-1)\psi'(x-1) - 1.$$

Substitute this in (5.2) to obtain
(5.15)
$$\sum_{n=1}^\infty B_n^{(2)*} z^n = -\frac{1}{2}\log z - \frac{1}{2}\left\{\psi\left(z + \frac{1}{z} - 1\right) + \left(z + \frac{1}{z} - 1\right)\psi'\left(z + \frac{1}{z} - 1\right) - 1\right\}.$$

This completes the proof.

6. Integrals involving Chebyshev polynomials

This section presents the evaluation of some integrals involving the Chebyshev polynomials obtained as byproducts of the former results. The proof uses the Binet formulas (1.13) and (1.14) for these polynomials. The discussion begins with some preliminary results.

Lemma 6.1. *Let $b > 0$. Then*

$$(6.1) \qquad \frac{d^{2\ell}}{du^{2\ell}} \log(1 + bu^2) = \frac{2(-1)^{\ell-1} b^\ell (2\ell - 1)!}{(1 + bu^2)^\ell} T_{2\ell} \left(\frac{1}{\sqrt{1 + bu^2}} \right)$$

and

$$(6.2) \qquad \frac{d^{2\ell+1}}{du^{2\ell+1}} \log(1 + bu^2) = \frac{2(-1)^\ell b^{\ell+1} (2\ell)! u}{(1 + bu^2)^{\ell+1}} U_{2\ell} \left(\frac{1}{\sqrt{1 + bu^2}} \right).$$

Proof. The proof is given for the second formula. The first one can be established by the same procedure. Successive differentiation gives

$$(6.3) \qquad \frac{d^{2\ell+1}}{du^{2\ell+1}} \log(1 \pm i\sqrt{b}u) = \pm \frac{(-1)^\ell i b^{\ell+1/2} (2\ell)!}{(1 \pm i\sqrt{b}u)^{2\ell+1}}.$$

Hence

$$\frac{d^{2\ell+1}}{du^{2\ell+1}} \log(1 + bu^2) = (-1)^\ell i b^{\ell+1/2} (2\ell)! \left\{ \frac{1}{(1 + i\sqrt{b}u)^{2\ell+1}} - \frac{1}{(1 - i\sqrt{b}u)^{2\ell+1}} \right\}$$

$$= \frac{2(-1)^\ell b^{\ell+1} (2\ell)! u}{(1 + bu^2)^{\ell+1}} U_{2\ell} \left(\frac{1}{\sqrt{1 + bu^2}} \right)$$

using (1.14). $\qquad\qquad\square$

The representation for the densities $\rho_\ell(u)$ given by Airault are now used to produce some spectacular integrals involving the Chebyshev polynomials.

Theorem 6.2. *Let $T_\ell(x)$ be the Chebyshev polynomial of the first kind. Define*

$$(6.4) \qquad P_1(u, \ell) = \prod_{j=1}^{\ell-1} (u^2 + j^2) \text{ and } P_2(u, \ell) = \prod_{j=1}^{\ell} \left(u^2 + \left(j - \tfrac{1}{2}\right)^2 \right)$$

Then, for $x > \ell$,

$$\int_0^\infty \left\{ \frac{u P_1(u, \ell)}{\tanh(\pi u)} - u^{2\ell-1} \right\} T_{2\ell} \left(\frac{x - \ell}{\sqrt{u^2 + (x - \ell)^2}} \right) \frac{du}{(u^2 + (x - \ell)^2)^\ell} =$$

$$(-1)^\ell \left(\log(x - \ell) + H_{2\ell-1} - \frac{d^{2\ell-1}}{dx^{2\ell-1}} \left\{ \binom{x-1}{2\ell-1} \psi(x - \ell) \right\} \right),$$

and for $x > \ell + \tfrac{1}{2}$,

$$\int_0^\infty \left\{ \tanh(\pi u) P_2(u, \ell) - u^{2\ell} \right\} U_{2\ell} \left(\frac{x - \ell - \tfrac{1}{2}}{\sqrt{u^2 + (x - \ell - \tfrac{1}{2})^2}} \right) \frac{u \, du}{(u^2 + (x - \ell - \tfrac{1}{2})^2)^{\ell+1}} =$$

$$(-1)^\ell \left(\log(x - \ell - \tfrac{1}{2}) + H_{2\ell} - \frac{d^{2\ell}}{dx^{2\ell}} \left\{ \binom{x-1}{2\ell} \psi(x - \ell - \tfrac{1}{2}) \right\} \right).$$

Proof. The details are given for the second formula. The expression for the density functions in given by Airault in (4.12) are written as

$$(6.5) \qquad \rho_{2\ell+1}(u) = C_\ell \left(\frac{d}{du} \right)^{2\ell+1} [P_2(u,\ell) \tanh(\pi u)]$$

with $C_\ell = (2(2\ell)!)^{-1}$. Therefore

$$\int_0^\infty \rho_{2\ell+1}(u) \log(1+bu^2)\, du = C_\ell \int_0^\infty \log(1+bu^2) \left(\frac{d}{du} \right)^{2\ell+1} [P_2(u,\ell) \tanh(\pi u)]\, du$$

$$= C_\ell \int_0^\infty \log(1+bu^2) \frac{d}{du} \left[\left(\frac{d}{du} \right)^{2\ell} [P_2(u,\ell) \tanh(\pi u)] \right]\, du.$$

In order to integrate by parts, the boundary terms at $+\infty$ need to be modified. Observe that

$$\left(\frac{d}{du} \right)^{2\ell} [P_2(u,\ell) \tanh(\pi u)] = \sum_{j=0}^{2\ell} \binom{2\ell}{j} \left(\frac{d}{du} \right)^j [\tanh(\pi u)] \left(\frac{d}{du} \right)^{2\ell-j} [P_2(u,\ell)]$$

$$= (2\ell)! \tanh(\pi u) +$$

$$\sum_{j=1}^{2\ell} \binom{2\ell}{j} \left(\frac{d}{du} \right)^j [\tanh(\pi u)] \left(\frac{d}{du} \right)^{2\ell-j} [P_2(u,\ell)]$$

The terms coming from derivatives of $\tanh(\pi u)$ in the second sum are polynomials in $\text{sech}^2 u$, without a constant term. The terms coming from $P_2(u,\ell)$ are polynomials in u. It follows that the whole second sum vanishes as $u \to +\infty$. Then

$$\int_0^\infty \rho_{2\ell+1}(u) \log(1+bu^2)\, du$$

$$= C_\ell \int_0^\infty \log(1+bu^2) \frac{d}{du} \left[\left(\frac{d}{du} \right)^{2\ell} \left[P_2(u,\ell) \tanh(\pi u) - u^{2\ell} \right] \right]\, du.$$

and now integration by parts gives

$$\int_0^\infty \rho_{2\ell+1}(u) \log(1+bu^2)\, du$$

$$= C_\ell \int_0^\infty \left[P_2(u,\ell) \tanh(\pi u) - u^{2\ell} \right] \left(\frac{d}{du} \right)^{2\ell+1} \log(1+bu^2)\, du.$$

Now use Theorem 1.3 to evaluate the integral on the left-hand side and Lemma 6.1 to obtain the result. $\qquad \square$

7. Relations to the Hurwitz and Barnes zeta functions

This section expresses the densities $\rho_\ell(x)$ in terms of the Hurwitz zeta function. This is the used to produce the closed-form evaluations of some integrals involving the Hurwitz zeta function.

Definition 7.1. Let $N \in \mathbb{N}$ and $w,\, s \in \mathbb{C}$ with $\operatorname{Re} w > 0$, $\operatorname{Re} s > N$. The *Barnes zeta function* is defined by the series

$$(7.1) \qquad \zeta_N(s, w | a_1, \cdots, a_N) = \sum_{m_1, \cdots, m_N = 0}^\infty (w + m_1 a_1 + \cdots + m_N a_N)^{-s}.$$

This function was introduced in [3] and contains, as the special case $N = 1$ and $a_1 = 1$, the *Hurwitz zeta function*

$$(7.2) \qquad \zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n + w)^s}.$$

A class of definite integrals connected to $\zeta(s, w)$ was described in [8, 9]. In particular, the classical identity of Lerch [11, entry 9.533.3]

$$(7.3) \qquad \frac{d}{dz} \zeta(z, q) \big|_{z=0} = \log \Gamma(q) - \log \sqrt{2\pi}$$

gives the classical evaluation

$$(7.4) \qquad \int_0^1 \log \Gamma(z) \, dz = \log \sqrt{2\pi}$$

given by L. Euler, as well as

$$(7.5) \quad \int_0^1 \log^2 \Gamma(z) \, dz = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3} \gamma \log \sqrt{2\pi} + \frac{4}{3} \left(\log \sqrt{2\pi} \right)^2$$
$$- (\gamma + 2\log \sqrt{2\pi}) \frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}.$$

The corresponding evaluations for the integrals of $\log^\ell \Gamma(z)$, for $\ell = 3, 4$ are more complicated and they involve multiple-zeta values. In particular, the existence of formulas for $\ell \geq 5$, remains an open problem. See [2] for details.

The connection between the Hurwitz zeta function and the densities $\rho_\ell(x)$ is based on an integral representation of the Barnes zeta function given by S. N. M. Ruijsenaars [14, p. 121]. Introducing the notation $A_M = \frac{1}{2}(a_1 + \cdots + a_M)$, it is shown in [14] that if $\delta > -A_M$, the Barnes zeta function has the integral representation

$$(7.6) \quad \zeta_M(s, A_M + \delta + iz | a_1, \cdots, a_M) = \frac{2^{1-M}}{\Gamma(s)} \int_0^\infty \frac{(2y)^{s-1} e^{-2\delta y}}{\prod_{1 \leq j \leq M} \sinh(a_j y)} e^{-2izy} dy$$

for $\operatorname{Re} s > M$ and $\operatorname{Im} z < A_M + \delta$. Now choose $n \in \mathbb{N}$ and consider the special case $\delta = 0$, $M = \ell$, $s = \ell + 1$ and $a_j = 1$ for $1 \leq j \leq M$. This yields the identity

$$(7.7) \qquad \zeta_\ell(\ell + 1, \tfrac{\ell}{2} + iz | 1, \cdots, 1) = \frac{2}{\ell!} \int_0^\infty e^{-2izy} \left(\frac{y}{\sinh y} \right)^\ell dy.$$

The next result gives a new representation for the density $\rho_\ell(x)$ in terms of the Barnes zeta function. The proof comes directly from (4.6).

Proposition 7.2. *Let*

$$(7.8) \qquad \zeta_\ell(m, z) = \zeta_\ell(m, z | 1, \cdots, 1).$$

Then the density function $\rho_\ell(x)$ is given by

$$(7.9) \qquad \rho_\ell(x) = \frac{\ell!}{2\pi} \left(\zeta_\ell(\ell + 1, \tfrac{\ell}{2} + ix) + \zeta_\ell(\ell + 1, \tfrac{\ell}{2} - ix) \right).$$

The next representation for the densities $\rho_\ell(x)$ comes from a result of J. Choi [5, Equation (2.5)], which expresses $\zeta_\ell(s, w)$ as a finite linear combination of the Hurwitz zeta function, in the form

$$(7.10) \qquad \zeta_\ell(s, w) = \sum_{j=0}^{\ell-1} p_{\ell,j}(w) \zeta(s - j, w)$$

where

$$(7.11) \qquad p_{\ell,j}(w) = \frac{(-1)^{\ell+1-j}}{(\ell-1)!} \sum_{m=j}^{\ell-1} \binom{m}{j} s(\ell, m+1) w^{m-j},$$

where $s(\ell, m)$ is the Stirling number of the first kind. Then (7.9) leads to

$$(7.12) \quad \rho_\ell(x) = \frac{\ell(-1)^{\ell+1}}{2\pi} \sum_{j=0}^{\ell-1} (-1)^j \sum_{m=j}^{\ell-1} \binom{m}{j} s(\ell, m+1)$$

$$\times \left\{ \left(\frac{\ell}{2}+ix\right)^{m-j} \zeta\left(\ell+1-j, \frac{\ell}{2}+ix\right) + \left(\frac{\ell}{2}-ix\right)^{m-j} \zeta\left(\ell+1-j, \frac{\ell}{2}-ix\right) \right\}.$$

It follows that the logarithmic moment can be expressed as

$$\int_0^\infty \rho_{2\ell}(u) \log\left(1 + \frac{u^2}{(x-\ell)^2}\right) du =$$

$$\frac{2\ell}{\pi} \sum_{j=0}^{\ell-1} (-1)^{j-1} \sum_{m=j}^{2\ell-1} s(2\ell, m+1)$$

$$\times \operatorname{Re} \int_0^\infty \left\{ (\ell+iu)^{m-j} \zeta(2\ell+1-j, \ell+iu) \right\} \log\left(1 + \frac{u^2}{(x-\ell)^2}\right) du.$$

Now replace ℓ by 2ℓ in (3.22) and Theorem 1.1, equate their right-hand sides, and use the above identity to arrive at first of the following two identities. The second one is similarly proved.

Theorem 7.3. *Let $\zeta(s, w)$ denote the Hurwitz zeta function and $s(\ell, m)$ the Stirling numbers of the first kind. Define*

$$z_\ell(m, j)(x) = 2 \operatorname{Re} \int_0^\infty (\ell+iu)^{m-j} \zeta(2\ell+1-j, \ell+iu) \log\left(1 + \frac{u^2}{(x-\ell)^2}\right) du$$

and

$$Z_\ell(m, j)(x) = 2 \operatorname{Re} \int_0^\infty (\ell+\tfrac{1}{2}+iu)^{m-j} \zeta(2\ell+2-j, \ell+\tfrac{1}{2}+iu) \log\left(1 + \frac{u^2}{(x-\ell-\frac{1}{2})^2}\right) du$$

Then, for $x > \ell$,

$$\sum_{j=0}^{2\ell-1} (-1)^{j-1} \sum_{m=j}^{2\ell-1} \binom{m}{j} s(2\ell, m+1) z_\ell(m, j)(x) =$$

$$-\frac{\pi}{\ell} \left(\log(x-\ell) + H_{2\ell-1} - \frac{d^{2\ell-1}}{dx^{2\ell-1}} \left\{ \binom{x-1}{2\ell-1} \psi(x-\ell) \right\} \right),$$

and $x > \ell + \frac{1}{2}$,

$$\sum_{j=0}^{2\ell} (-1)^j \sum_{m=j}^{2\ell} \binom{m}{j} s(2\ell+1, m+1) Z_\ell(m, j)(x) =$$

$$-\frac{2\pi}{2\ell+1} \left(\log\left(x-\ell-\tfrac{1}{2}\right) + H_{2\ell} - \frac{d^{2\ell}}{dx^{2\ell}} \left\{ \binom{x-1}{2\ell} \psi\left(x-\ell-\tfrac{1}{2}\right) \right\} \right).$$

Inverting these systems of equations to obtain expressions for $Y_n(m, j)$ and $Z_n(m, j)$ is an open problem.

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