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THE LAPLACE TRANSFORM OF THE DIGAMMA FUNCTION: AN INTEGRAL DUE TO GLASSER, MANNA AND OLOA

TEWODROS AMDEBERHAN, OLIVIER ESPINOSA, AND VICTOR H. MOLL

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Abstract. The definite integral

$$M(a) := \frac{4}{\pi}\int_0^{\pi/2} \frac{x^2\,dx}{x^2+\ln^2(2e^{-a}\cos x)}$$

is related to the Laplace transform of the digamma function

$$L(a) := \int_0^\infty e^{-as}\psi(s+1)\,ds,$$

by $M(a) = L(a) + \gamma/a$ when $a > \ln 2$. Certain analytic expressions for $M(a)$ in the complementary range, $0 < a \leq \ln 2$, are also provided.

1. Introduction

The classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik [7] contains a large collection organized in sections according to the form of the integrand. In each section one finds a significant variation on the complexity of the integrals. For example, section 4.33–4.34, with the title *Combinations of logarithms and exponentials*, presents the elementary formula 4.331.1. For $a > 0$,

$$\int_0^\infty e^{-ax}\ln x\,dx = -\frac{\gamma+\ln a}{a}, \tag{1.1}$$

where γ is the *Euler constant*

$$\gamma = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} - \ln n. \tag{1.2}$$

Contained in the same section are the more elaborate 4.332.1 and 4.325.6:

$$\int_0^\infty \frac{\ln x\,dx}{e^x+e^{-x}-1} = \int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{x^2-x+1} = \frac{2\pi}{\sqrt{3}}\left(\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right).$$

The difficulty involved in the evaluation of a definite integral is hard to measure from the complexity of the integrand. For instance, the evaluation of *Vardi's integral*,

$$\int_{\pi/4}^{\pi/2}\ln\ln\tan x\,dx = \int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\left(\frac{\Gamma(\frac{3}{4})\sqrt{2\pi}}{\Gamma(\frac{1}{4})}\right), \tag{1.3}$$

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that appears as 4.229.7 in [7], requires a reasonable amount of number theory. The second integral form is 4.325.4, found in the section entitled *Combinations of logarithmic functions of more complicated arguments and powers*. The reader will find in [15] a discussion of this formula.

It is a remarkable fact that combinations of elementary functions in the integrand often exhibit definite integrals whose evaluation is far from elementary. A systematic study of the formulas in [7] has been initiated in the series [1, 2, 9, 10, 11, 12]. These papers are organized according to the *combinations* appearing in the integrand. Even the elementary cases, such as the combination of logarithms and rational functions discussed in [2], entail interesting results. The evaluation

$$\begin{aligned} \int_0^b \frac{\ln t \, dt}{(1+t)^{n+1}} &= \frac{1}{n}\left[1-(1+b)^{-n}\right]\ln b - \frac{1}{n}\ln(1+b) \\ &- \frac{1}{n(1+b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!}\binom{n-1}{j} |s(j+1,2)| b^j, \end{aligned} \tag{1.4}$$

for $b > 0$ and $n \in \mathbb{N}$, produces an explicit formula for the case where the rational function has a single pole. Here, $|s(n,k)|$ are the *unsigned Stirling numbers of the first kind*, which count the number of permutations of n letters having exactly k cycles. The case of a purely imaginary pole,

$$\int_0^x \frac{\ln t \, dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left[g_0(x) + p_n(x)\ln x - \sum_{k=0}^{n-1} \frac{\tan^{-1} x + p_k(x)}{2k+1} \right],$$

is expressed in terms of the rational function

$$p_n(x) = \sum_{j=1}^{n} \frac{2^{2j}}{2j\binom{2j}{j}} \frac{x}{(1+x^2)^j}, \tag{1.5}$$

and with

$$g_0(x) = \ln x \, \tan^{-1} x - \int_0^x \frac{\tan^{-1} t}{t} \, dt. \tag{1.6}$$

The special case $x = 1$ becomes

$$\int_0^1 \frac{\ln t \, dt}{(1+t^2)^{n+1}} = -2^{-2n}\binom{2n}{n}\left(G + \sum_{k=0}^{n-1} \frac{\frac{\pi}{4} + p_k(1)}{2k+1} \right), \tag{1.7}$$

where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{1.8}$$

is *Catalan's constant*. The values

$$p_k(1) = \sum_{j=1}^{k} \frac{2^j}{2j\binom{2j}{j}} \tag{1.9}$$

do not admit a closed form (in the sense of [14]), but they do satisfy the three-term recurrence

$$(2k+1)p_{k+1}(1) - (3k+1)p_k(1) + kp_{k-1}(1) = 0. \tag{1.10}$$

The study of definite integrals, where the integrand is a combination of powers, logarithms and trigonometric functions, was initiated by Euler [5], with the evaluation of

$$\int_0^{\pi/2} x \, \ln(2\cos x)\, dx = -\frac{7}{16}\zeta(3) \tag{1.11}$$

and

$$\int_0^{\pi/2} x^2 \, \ln(2\cos x)\, dx = -\frac{\pi}{4}\zeta(3), \tag{1.12}$$

which appear in his study of the *Riemann zeta* function at the odd integers. These type of integrals have been investigated in [8], [16]. The *intriguing integral* of D. and J. Borwein [3],

$$\int_0^{\pi/2} x^2 \, \ln^2(2\cos x)\, dx = \frac{11\pi}{16}\zeta(4) = \frac{11\pi^5}{1440}, \tag{1.13}$$

was first conjectured on the basis of a numerical computation by Enrico Au-Yueng while he was an undergraduate student at the University of Waterloo. A nice example of experimental mathematics in action.

Recently O. Oloa considered the integral

$$M(a) := \frac{4}{\pi}\int_0^{\pi/2} \frac{x^2\, dx}{x^2 + \ln^2(2e^{-a}\cos x)}, \tag{1.14}$$

and the special value

$$M(0) = \frac{4}{\pi}\int_0^{\pi/2} \frac{x^2\, dx}{x^2 + \ln^2(2\cos x)} = \frac{1}{2}(1 + \ln(2\pi) - \gamma) \tag{1.15}$$

is established in [13].

Oloa's method of proof relies on the expansion

$$\frac{x^2}{x^2 + \ln^2(2\cos x)} = x\sin 2x + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{a_n}{n!} - \frac{a_{n+1}}{(n+1)!}\right) x\sin(2nx). \tag{1.16}$$

Here

$$a_n := \int_0^1 (t)_n \, dt, \tag{1.17}$$

where $(t)_n = t(t+1)\cdots(t+n-1)$ is the *Pochhammer symbol*. The standard relation

$$(t)_n = \sum_{k=1}^{n} |s(n,k)| t^k \tag{1.18}$$

gives

$$a_n = \sum_{k=1}^{n} \frac{|s(n,k)|}{k+1}. \tag{1.19}$$

M. L. Glasser and D. Manna [6] introduced the function

$$L(a) := \int_0^{\infty} e^{-as}\psi(s+1)\, ds, \tag{1.20}$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the *digamma function*. After integrating by parts and making use of (1.1), one finds

$$L(a) = -\gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t)\, dt. \tag{1.21}$$

The main result in [6] gives a relation between $M(a)$ and $L(a)$.

Theorem 1.1. *If $a > \ln 2$, then*

$$M(a) = L(a) + \frac{\gamma}{a}.$$

That is, for $a > \ln 2$,

$$M(a) = \frac{\gamma}{a} - \gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t)\, dt. \tag{1.22}$$

The proof in [6] begins with the representation

$$\int_0^{\pi/2} \cos^\nu x \, \cos ax \, dx = \frac{\pi \Gamma(\nu+2)}{2^{\nu+1}(\nu+1)\,\Gamma(1+\frac{\nu}{2}+\frac{a}{2})\,\Gamma(1+\frac{\nu}{2}-\frac{a}{2})} \tag{1.23}$$

(3.631.9 in [7]). Differentiating with respect to a, evaluating at $a = s$, and using $\psi(1) = -\gamma$ yields

$$\psi(s+1) = \frac{2^{s+2}}{\pi} \int_0^{\pi/2} x \cos^s x \, \sin(sx)\, dx - \gamma. \tag{1.24}$$

Replacing (1.24) into (1.20) produces

$$L(a) + \frac{\gamma}{a} = -\frac{4}{\pi} \operatorname{Im} \int_0^\infty \int_0^{\pi/2} x e^{s(\ln\left[2e^{-a}\cos x\right] - ix)}\, dx\, ds. \tag{1.25}$$

The identity (1.22) follows from evaluating the s-integral as

$$\int_0^\infty e^{s(\ln\left[2e^{-a}\cos x\right] - ix)}\, ds = \frac{1}{ix - \ln\left[2e^{-a}\cos x\right]}. \tag{1.26}$$

The authors of [6] produced a series expansion for $M(a)$, which they recognize as a hypergeometric function in two variables, and state that *this strongly suggests that for a general value of a, no further progress is possible.* The hypergeometric expression gives

$$M(0) = 1 + \frac{1}{2}\int_0^1 t(1-t)\, {}_3F_2(1,1,2-t;2,3;1)\, dt, \tag{1.27}$$

on which they invoke

$${}_3F_2(1,1,2-t;2,3;1) = \frac{2(1-\gamma-\psi(t+1))}{1-t} \tag{1.28}$$

to give a new proof of (1.15).

The graph of $M(a)$ shown in Figure 1, obtained by the numerical integration of (1.14), has a well-defined *cusp* at $a = \ln 2$. In this paper, analytic expressions for both branches of $M(a)$ are provided. The region $a > \ln 2$, determined in [6], has been reviewed in the present section. The corresponding expressions for $0 < a < \ln 2$ is the content of the next section.

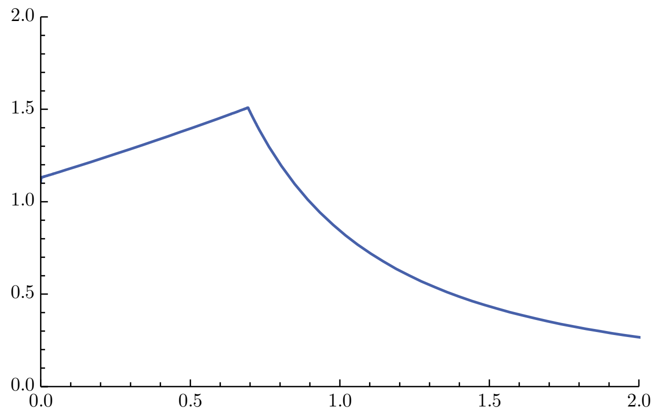

FIGURE 1. The graph of $M(a)$ for $0 \leq a \leq 2$

2. THE CASE $0 < a < \ln 2$

The representation

$$M(a) = -\frac{e^a}{2\pi} \operatorname{Im} \int_0^1 e^{-at} \int_{-\pi}^{\pi} \frac{x(1+e^{ix})^t}{1-e^a+e^{ix}} \, dx \, dt \tag{2.1}$$

was established in [6]. Their proof is replicated here for the sake of the reader's convenience. The identity

$$\operatorname{Im} \frac{x}{ix + \ln\left[2e^{-a}\cos x\right]} = \frac{x^2}{x^2 + \ln^2\left[2e^{-a}\cos x\right]} \tag{2.2}$$

yields

$$M(a) = \frac{4}{\pi} \operatorname{Im} \int_0^{\pi/2} \frac{x \, dx}{ix + \ln\left[2e^{-a}\cos x\right]}. \tag{2.3}$$

If $a > \ln 2$, then

$$\int_0^{\infty} e^{s\ln\left[2e^{-a}\cos x\right] + ix} \, ds = \frac{1}{ix + \ln\left[2e^{-a}\cos x\right]}. \tag{2.4}$$

This implies

$$M(a) = \frac{2}{\pi} \operatorname{Im} \int_{-\pi/2}^{\pi/2} \int_0^{\infty} x e^{s(\ln\left[2e^{-a}\cos x\right] + ix)} \, dx \, ds, \tag{2.5}$$

where one uses the fact that the imaginary part of the integrand is an *even* function of x. One more identity,

$$e^{isx} \cdot e^{s\ln\left[2e^{-a}\cos x\right]} = e^{s\ln\left[e^{-a}(1+e^{2ix})\right]}, \tag{2.6}$$

and the change of variables $x \mapsto x/2$, give the equality

$$M(a) = \frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \int_0^{\infty} x e^{s\ln\left[e^{-a}(1+e^{ix})\right]} \, ds \, dx. \tag{2.7}$$

Evaluating the s-integral yields

$$M(a) = -\frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \frac{x \, dx}{\ln\left[e^{-a}(1+e^{ix})\right]}. \tag{2.8}$$

The formula

$$\frac{1}{\ln u} = \int_0^1 \frac{u^t\,dt}{u-1} \tag{2.9}$$

now gives (2.1) from (2.8).

Note 2.1. The proof outlined above is valid for $a > \ln 2$, but (2.1) holds for $a > 0$.

Notation. Define $b := e^a - 1$ and let $0 < a < \ln 2$ so that $0 < b < 1$.

The terms $(1+e^{ix})^t$ and $1/(1-be^{-ix})$ from (2.1) are now expanded in a power series to produce

$$M(a) = -\frac{e^a}{2\pi}\int_0^1\int_{-\pi}^{\pi} xe^{-at}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b^j\binom{t}{k}\sin[x(k-j-1)]\,dx\,dt.$$

The term corresponding to $k = j+1$ disappears, and a computation of the x-integral gives

$$\begin{aligned} M(a) \quad &= \quad e^a\int_0^1 e^{-at}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j-k}\frac{b^j\binom{t}{k}}{j+1-k}\,dt \\ &+ \quad e^a\int_0^1 e^{-at}\sum_{j=1}^{\infty} b^j\sum_{\nu=1}^{j}\frac{(-1)^\nu}{\nu}\binom{t}{\nu+j}\,dt. \end{aligned} \tag{2.10}$$

Lemma 2.1. *Let $t \in \mathbb{R}$ and $j \in \mathbb{N}\cup\{0\}$. Then*

$$\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}\binom{t}{\nu+j} = \binom{t}{j}\left[\psi(j+1)-\psi(t+1)\right]. \tag{2.11}$$

Proof. The integral representation (3.268.2 in [7])

$$\psi(p+1)-\psi(q+1) = -\int_0^1 \frac{x^p-x^q}{1-x}\,dx \tag{2.12}$$

yields

$$\psi(p+1)-\psi(q+1) = \sum_{j=1}^{\infty}(-1)^{j-1}\left(\binom{p}{j}-\binom{q}{j}\right). \tag{2.13}$$

The result now follows from the identity

$$\binom{t}{k}^{-1}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\binom{t}{m+k} - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\binom{t}{m} = \sum_{m=1}^{k}\frac{(-1)^{m-1}}{m}\binom{k}{m}. \tag{2.14}$$

Apply the difference operator $\Delta a(k) := a(k+1)-a(k)$ and use

$$\binom{t}{k+1}^{-1}\binom{t}{m+k+1} - \binom{t}{k}^{-1}\binom{t}{m+k} = -\frac{m}{k+1}\binom{t}{k+1}^{-1}\binom{t+1}{m+k+1}$$

to write the derived equation as

$$-\frac{\binom{t}{k+1}^{-1}}{k+1}\sum_{m=1}^{\infty}(-1)^m\binom{t+1}{m+k+1} = \Delta\sum_{m=1}^{k}\frac{(-1)^m}{m}\binom{k}{m}. \tag{2.15}$$

The left hand side of (2.14) reduces to $-1/(k+1)$ in view of the classical identity

$$\sum_{m=1}^{\infty}(-1)^{m-1}\binom{t+1}{m+k+1} = \binom{t}{k+1}. \tag{2.16}$$

A simple evaluation of the right hand side in (2.14) also produces $-1/(k+1)$. Therefore both sides of (2.14) are, up to a constant, the *harmonic number* H_k. The special case $k=0$ shows that this constant vanishes. □

Continuing from (2.10), it follows that

(2.17)

$$\begin{aligned} M(a) &= e^a\int_0^1 \sum_{j=0}^{\infty} b^j \sum_{k=0}^{j} \frac{(-1)^{j-k}\binom{t}{k}}{j+1-k} + \frac{e^a}{b}\int_0^1 e^{-at}\sum_{j=1}^{\infty} b^j\binom{t}{j}\psi(j+1)\,dt \\ &- \frac{e^a}{b}\int_0^1 e^{-at}\sum_{j=1}^{\infty} b^j\binom{t}{j}\psi(t+1)\,dt \equiv M_1+M_2+M_3. \end{aligned}$$

To simplify M_1, observe

$$\begin{aligned} \sum_{j=0}^{\infty} b^j \sum_{k=0}^{j} \frac{(-1)^{j-k}\binom{t}{k}}{j+1-k} &= \sum_{k=0}^{\infty} b^k\binom{t}{k}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}b^{\nu}}{\nu+1} \\ &= \frac{\ln(1+b)}{b}\sum_{k=0}^{\infty}\binom{t}{k}b^k = \frac{ae^{at}}{b}. \end{aligned}$$

Thus, $M_1 = a/(1-e^{-a})$.

The reduction of M_2 employs the following result.

Lemma 2.2. *If* $0<a<\ln 2$*, then*

$$\int_0^1 e^{-at}\sum_{j=0}^{\infty} b^j\binom{t}{j}\psi(j+1)\,dt = \ln(1-e^{-a}) + \int_1^{\infty}\frac{e^{-at}}{t}\,dt. \tag{2.18}$$

Proof. The Stirling numbers $s(j,k)$ satisfy

$$j!\binom{t}{j} = \sum_{k=0}^{j} s(j,k)t^k, \tag{2.19}$$

so that

$$\int_0^1 e^{-at}\sum_{j=0}^{\infty} b^j\binom{t}{j}\psi(j+1)\,dt = \frac{e^{-a}b\gamma}{a} + e^{-a}\sum_{j=1}^{\infty}(b^{j+1}\alpha_j - b^j\alpha_{j-1})\psi(j+1), \tag{2.20}$$

with

$$\alpha_j(a) = \frac{1}{j!}\sum_{k=0}^{j}\frac{s(j,k)k!}{a^{k+1}}. \tag{2.21}$$

The result now follows from integration by parts and the identity

$$\sum_{j=k}^{\infty}\frac{s(j,k)b^j}{j!} = \frac{\ln^k(1+b)}{k!}. \tag{2.22}$$

□

Therefore,

$$M_2 = \frac{\ln(1-e^{-a})}{1-e^{-a}} + \frac{1}{1-e^{-a}} \int_1^\infty \frac{e^{-at}}{t}\,dt. \tag{2.23}$$

Finally,

$$M_3 = -\frac{e^a}{b}\int_0^1 e^{-at}\left(\sum_{j=1}^\infty \binom{t}{j} b^j\right)\psi(t+1)\,dt = -\frac{e^a}{b}\int_0^1 (1-e^{-at})\psi(t+1)\,dt.$$

A direct computation shows that $\int_0^1 \psi(t+1)\,dt = 0$, and integration by parts gives

$$M_3 = \frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t+1)\,dt. \tag{2.24}$$

The identity $\ln\Gamma(t+1) = \ln\Gamma(t) + \ln t$ now yields

$$M_3 = \frac{a}{1-e^{-a}}\left(\int_0^1 e^{-at}\ln t\,dt + \int_0^1 e^{-at}\ln\Gamma(t)\,dt\right).$$

Replacing (2.17), (2.19) and (2.21) into (2.17) provides an expression for $M(a)$:

$$\begin{aligned} M(a) &= \frac{a}{1-e^{-a}} + \frac{\gamma}{a} + \frac{\ln(1-e^{-a})}{1-e^{-a}} \\ &+ \frac{a}{1-e^{-a}}\int_0^\infty e^{-at}\ln t\,dt + \frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t)\,dt. \end{aligned} \tag{2.25}$$

The term γ/a comes from the index $j=0$ in the sum (2.18). The main result presented here now follows from (1.1). This settles a conjecture of O. Oloa presented in [13].

Theorem 2.1. *If $0 < a < \ln 2$, then*

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) - \gamma - \ln a}{1-e^{-a}} + \frac{a}{1-e^{-a}}\int_0^1 e^{-at}\ln\Gamma(t)\,dt.$$

The above result is complementary to Theorem 1.1.

Corollary 2.1. *If $0 < a < \ln 2$, then*

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) + \Gamma(0,a)}{1-e^{-a}} + \frac{1}{1-e^{-a}}\int_0^1 e^{-at}\psi(t+1)\,dt,$$

where $\Gamma(0,a)$ is the incomplete gamma function.

Proof. Split up the first integral in (2.25) and integrate by parts. □

The derivative of (2.1) at $a=0$, the classical values

$$\int_0^1 \ln\Gamma(t)\,dt = \frac{1}{2}\ln 2\pi \tag{2.26}$$

and

$$\int_0^1 t\ln\Gamma(t)\,dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6}\ln 2\pi - \frac{\gamma}{12}, \tag{2.27}$$

obtained in [4], give

$$\int_0^{\pi/2} \frac{x^2\ln(2\cos x)\,dx}{(x^2+\ln^2(2\cos x))^2} = \frac{7\pi}{192} + \frac{\pi\,\ln 2\pi}{96} - \frac{\zeta'(2)}{16\pi}. \tag{2.28}$$

Further differentiation of (2.1) produces the evaluation of a family of integrals similar to (2.28).

The integral in (2.1) can be expressed in an alternative form. Define

$$\Lambda(z) := \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{j}{j^2+z^2}-\ln n\right). \tag{2.29}$$

Observe that $\Lambda(0)=\gamma$, so $\Lambda(z)$ is a generalization of Euler's constant.

Lemma 2.3. *Let $a>0$, $c=1-e^{-a}$ and define $A:=\ln 2\pi+\gamma$. Then*

$$\int_0^1 e^{-at}\ln\Gamma(t)\,dt = \frac{A(a-c)}{a^2}-\frac{c}{2a}\Lambda\left(\frac{a}{2\pi}\right)+2c\sum_{j=1}^{\infty}\frac{\ln j}{a^2+4\pi^2j^2}. \tag{2.30}$$

Proof. Expand the exponential into a MacLaurin series and use the value

$$\begin{aligned}\int_0^1 t^n\ln\Gamma(t)\,dt &= \frac{1}{n+1}\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^k\binom{n+1}{2k-1}\frac{(2k)!}{k(2\pi)^{2k}}\left[A\zeta(2k)-\zeta'(2k)\right]\\ &- \frac{1}{n+1}\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n+1}{2k}\frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1)+\frac{\ln\sqrt{2\pi}}{n+1}\end{aligned}$$

given as (6.14) in [4]. Then interchange the resulting double sums. □

The next corollary follows from the identity $M(a)=L(a)+\gamma/a$.

Corollary 2.2. *If $0<a<\ln 2$ and $c=1-e^{-a}$, then*

$$\int_0^{\infty} e^{-at}\ln\Gamma(t)\,dt = -\frac{\gamma+\ln a}{ace^a}+\frac{A(a-c)}{a^2c}-\frac{1}{2a}\Lambda\left(\frac{a}{2\pi}\right)+2\sum_{j=1}^{\infty}\frac{\ln j}{a^2+4\pi^2j^2}. \tag{2.31}$$

Lemma 2.4. *Let $f(t)=2^{-t}\ln\Gamma(t)$. Then*

$$\int_0^{\infty} f(t)\,dt = 2\int_0^1 f(t)\,dt-\frac{\gamma+\ln\ln 2}{\ln 2}, \tag{2.32}$$

$$\int_0^{\infty} tf(t)\,dt = 2\int_0^1 (t+1)f(t)\,dt-\frac{(\gamma+\ln\ln 2)(1+2\ln 2)-1}{\ln^2 2}. \tag{2.33}$$

Proof. The function $f(t)$ satisfies $f(t+1)=\frac{1}{2}f(t)+\frac{1}{2}2^{-t}\ln t$. Splitting the integral

$$\int_0^{\infty} f(t)\,dt = \int_0^1 f(t)\,dt+\int_0^{\infty} f(t+1)\,dt \tag{2.34}$$

and using (1.1) gives the first result. The proof of (2.33) is similar; it only requires differentiating (1.1) with respect to the parameter a. □

The reader will check that (2.32) is equivalent to the continuity of $M(a)$ at $a=\ln 2$. The identity (2.33) provides a proof of the next theorem, which in itself is worthy of singular (pun intended) interest.

Theorem 2.2. *The jump of $M'(a)$ at $a=\ln 2$ is 4.*

3. Conclusions

The integral

$$M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2\,dx}{x^2 + \ln^2(2e^{-a}\cos x)}$$

satisfies

$$M(a) = \frac{\gamma}{a} + \int_0^\infty e^{-at}\psi(t+1)\,dt \tag{3.1}$$

for $a > \ln 2$ and

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) + \Gamma(0,a)}{1-e^{-a}} + \frac{1}{1-e^{-a}} \int_0^1 e^{-at}\psi(t+1)\,dt$$

for $0 < a \leq \ln 2$.

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Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail address: tamdeber@tulane.edu

Departmento de Física, Universidad Téc. Federico Santa María, Valparaiso, Chile
E-mail address: olivier.espinosa@usm.cl

Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail address: vhm@math.tulane.edu