RAMANUJAN MASTER THEOREM APPLIED TO THE EVALUATION OF FEYNMAN DIAGRAMS

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Abstract. Ramanujan Master Theorem is a technique developed by S. Ramanujan to evaluate a class of definite integrals. This technique is used here to produce the values of integrals associated with Feynman diagrams.

1. Introduction

Precise experimental measurements in high energy physics require, in its theoretical counterpart, the development of new techniques for the evaluation of analytic objects associated with the corresponding Feynman diagrams. These techniques have lately emphasized the automatization of calculations of multiscale, multiloop diagrams.

Modern numerical methods for the evaluation of Feynman diagrams benefit from analytical tecniques employed as preliminary work to detect the presence of divergences. Recent advances include a method based on the Bernstein-Tkachov theorem for the corrections of one and two loop diagrams and methods based on sector-decompositions. New analytic methods to reduce Feynman diagrams to a small number of scalar integrals include integration by parts, the use of Lorenz invariance and other symmetries, Mellin-Barnes transforms and differential equations. The reader is referred to [9] for a description of these and other methods for the evaluation of Feynman diagrams and to [10, 11] for readable introductions to the topic.

This paper contains examples of an alternative method for the evaluation of some Feynman diagrams. It is based on the classical Ramanujan Master Theorem (RMT), one of his favorite techniques to evaluate definite integrals. The theoretical aspects of this method are presented in [7] and a collection of examples and some justification of the algorithm is given in [1, 3, 4]. This technique has also been used in [5] for the evaluation of some multidimensional integrals obtained by the Schwinger parametrization of Feynman diagrams.

The goal of the present work is to illustrate the flexibility of the method by evaluating integrals associated to two and three loop diagrams. Naturally the method works for a large variety of definite integrals and the first example illustrates this by computing the Mellin transform of a Bessel function. Further applications will be described in future work. Progress had been made in the automatization of the rules that control this method [8].

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2. Ramanujan's Master Theorem (RMT) and its generalization

2.1. The formalism. The Mellin transform

$$\mathcal{M}(f) = \int_0^\infty x^{\nu-1} f(x)\, dx \tag{2.1}$$

may be evaluated by one of Ramanujan's favorite tools; the so-called Ramanujan Master Theorem. It states that if $f(x)$ admits a series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-x)^n}{n!} \tag{2.2}$$

in a neighborhood of $x = 0$, with $f(0) = \varphi(0) \neq 0$, then

$$\int_0^\infty x^{\nu-1} f(x)\, dx = \Gamma(\nu)\varphi(-\nu). \tag{2.3}$$

The term $\varphi(-\nu)$ appearing in (2.3) requires an extension of the function φ, initially defined only for $\nu \in \mathbb{N}$. Details on the natural unique extension of φ are given in [1]. The condition $\varphi(0) \neq 0$ guarantees the convergence of the integral near $x = 0$, when $\nu > 0$. The proof of Ramanujan Master Theorem and the precise conditions for its application appear in [7].

2.2. The Mellin transform of a Bessel function. The first example computes an integral involving the Bessel function, with hypergeometric representation

$$J_\alpha\left(\sqrt{x}\right) = \left(\frac{\sqrt{x}}{2}\right)^\alpha \frac{1}{\Gamma\left(1+\alpha\right)} \, {}_0F_1\left(\left. \begin{matrix} \\ 1+\alpha \end{matrix} \right| -\frac{x}{4} \right), \tag{2.4}$$

with

$${}_0F_1\left(\left. \begin{matrix} \\ a \end{matrix} \right| x \right) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{x^n}{n!}, \tag{2.5}$$

and

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.6}$$

is the Pochhammer symbol. The integral evaluated here

$$I = \int_0^\infty x^{\beta-1} J_\alpha\left(\sqrt{x}\,\right)\, dx \tag{2.7}$$

is expressed as

$$\begin{aligned} I &= \int_0^\infty x^{\beta-1} \left(\frac{\sqrt{x}}{2}\right)^\alpha \frac{1}{\Gamma\left(1+\alpha\right)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\left(1+\alpha\right)_n} \frac{x^n}{4^n}\, dx \\ &= \int_0^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{1}{2^{\alpha+2n}\, \Gamma\left(1+\alpha+n\right)} \right] x^{n+\left(\beta+\frac{\alpha}{2}\right)-1}\, dx. \end{aligned} \tag{2.8}$$

In the notation of (2.2)

$$\varphi(n) = \frac{1}{2^{\alpha+2n}\, \Gamma(1+\alpha+n)}. \tag{2.9}$$

Therefore

$$I = \frac{\Gamma(n^*)}{2^{\alpha+2n^*}\Gamma(1+\alpha-n^*)}. \tag{2.10}$$

Here $n^* = -\left(\beta + \frac{\alpha}{2}\right)$, is the solution of

$$n + \beta + \tfrac{\alpha}{2} = 0. \tag{2.11}$$

Therefore

$$\int_0^\infty x^{\beta-1} J_\alpha\left(\sqrt{x}\right)\, dx = 2^{2\beta} \frac{\Gamma\left(\beta + \frac{\alpha}{2}\right)}{\Gamma\left(1 + \dfrac{\alpha}{2} - \beta\right)}. \tag{2.12}$$

This is entry 6.561.14 in the table of integrals [6].

2.3. **A second example: the Feynman diagram of a bubble.** This is the evaluation a D-dimensional integral corresponding to the massless bubble Feynman diagram. The result is well-known [2]. In momentum space the corresponding integral is given by

$$G := \int \frac{1}{i\pi^{D/2}} \frac{1}{[q^2]^{a_1}\left[(p-q)^2\right]^{a_2}}\, d^D q, \tag{2.13}$$

where the parameters $\{a_i\}$ are arbitrary. The Schwinger representation[1] corresponding to this diagram produces

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1}\, \frac{\exp\left(-\dfrac{xy}{x+y}\, p^2\right)}{(x+y)^{\frac{D}{2}}}\, dx\, dy. \tag{2.14}$$

In order to apply RMT in iterative form, each term of the integrand is expanded in a Taylor series. In situations where options are available, the optimal course of action seems to be to minimize the number of expansions. This is a heuristic rule and its justification is an open question. In this example, it is convenient to expand first the exponential function

$$\exp\left(-\frac{xy}{x+y}\, p^2\right) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^n}, \tag{2.15}$$

to produce

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^{\frac{D}{2}+n}}\, dx\, dy. \tag{2.16}$$

The next step is to expand $(x+y)^{-D/2-n}$ by the binomial theorem

$$(x+y)^{-(D/2+n)} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left(\tfrac{D}{2} + n\right)_k x^{-D/2-n-k} y^k \tag{2.17}$$

and replace in (2.16) to obtain

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)} \int_0^\infty \int_0^\infty \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\tfrac{D}{2} + n\right)_k\, x^{-k+a_1-\frac{D}{2}} y^{k+n+a_2} \frac{dx}{x} \frac{dy}{y}.$$

[1] There is a canonical procedure to associate to each Feynman diagram a multi-dimensional integral. For details, the reader is referred to [9, chapter 3], under the name *alpha parameters*.

Th change of variables $x \mapsto 1/x$ produces the alternative expression

$$G=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^\infty\int_0^\infty\sum_{k=0}^\infty\sum_{n=0}^\infty\frac{(-1)^n}{n!}\frac{(-1)^k}{k!}\left(p^2\right)^n\left(\tfrac{D}{2}+n\right)_k\;x^{k-a_1+\frac{D}{2}}y^{k+n+a_2}\frac{dx}{x}\frac{dy}{y}.$$

There are several options to employ RMT to evaluate this integral. Option a) evaluates first the integral in the x-variable using the expansion in the index k:

$$\int_0^\infty\sum_{k=0}^\infty\cdots\frac{(-x)^k}{k!}\,dx.$$

The value of the integral obtained by this procedure is denoted by G_a. The other two options, labeled G_b and G_c, are produced by replacing the pair (x,k) by (y,k) and (y,n), respectively. It is shown here that each of these options produces the same result.

Solution with option (a). In this case, G is given by

$$G_a=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^\infty\left[\int_0^\infty\frac{dx}{x}\sum_{k=0}^\infty\frac{(-1)^k}{k!}\varphi\left(k\right)\;x^{k-a_1+\frac{D}{2}}\right]\frac{dy}{y},\tag{2.18}$$

where $\varphi(k)$ is

$$\varphi\left(k\right)=\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(p^2\right)^n\left(\tfrac{D}{2}+n\right)_k\;y^{k+n+a_2}.\tag{2.19}$$

Ramanujan Master Theorem now gives

$$G_a=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^\infty\Gamma(k^*)\varphi\left(-k^*\right)\frac{dy}{y},\text{ with }k^*=\frac{D}{2}-a_1.$$

Thus,

$$\begin{aligned}G_a&=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\Gamma(D/2-a_1)\int_0^\infty\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(p^2\right)^n\left(\frac{D}{2}+n\right)_{a_1-\frac{D}{2}}\;y^{a_1+a_2-\frac{D}{2}+n}\frac{dy}{y}\\&=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\Gamma(D/2-a_1)\int_0^\infty\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(p^2\right)^n\frac{\Gamma\left(a_1+n\right)}{\Gamma\left(\frac{D}{2}+n\right)}y^{n+a_1+a_2-\frac{D}{2}}\frac{dy}{y}.\end{aligned}$$

The last integral is now evaluated using RMT to obtain

$$G_a=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\Gamma(\tfrac{D}{2}-a_1)\Gamma(n^*)\;\left(p^2\right)^{-n^*}\frac{\Gamma\left(a_1-n^*\right)}{\Gamma\left(\frac{D}{2}-n^*\right)},$$

with $n^*=a_1+a_2-\frac{D}{2}$. Therefore, option (a) gives the value of G as

$$G_a=(-1)^{-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-a_1-a_2}\frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_1)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma(a_1)\Gamma(a_2)\Gamma\left(D-a_1-a_2\right)}.\tag{2.20}$$

Solution with option (b). A similar argument now yields

$$G_b=\frac{(-1)^{-\frac{D}{2}}}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^\infty\left[\int_0^\infty\sum_{k=0}^\infty\frac{(-1)^k}{k!}\varphi\left(k\right)\;y^{k+n+a_2}\right]\frac{dy}{y}\frac{dx}{x},$$

with

$$\varphi(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \left(\tfrac{D}{2}+n\right)_k \; x^{k-a_1+\frac{D}{2}}.$$

Therefore

$$G_b \;=\; \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\,\Gamma(a_2)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma\left(\frac{D}{2}+n\right)} \; x^{-n-a_2-a_1+\frac{D}{2}} \frac{dx}{x}.$$

The change of variables $x \mapsto 1/x$ now gives

$$G_b = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\,\Gamma(a_2)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma\left(\frac{D}{2}+n\right)} \; x^{n+a_2+a_1-\frac{D}{2}} \frac{dx}{x}$$

and RMT produces the final result as

$$G_b = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2}-a_2-a_1} \frac{\Gamma(\frac{D}{2}-a_1)\Gamma\left(\frac{D}{2}-a_2\right)\Gamma(a_2+a_1-\frac{D}{2})}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(D-a_2-a_1)}. \tag{2.21}$$

Observe that $G_a = G_b$. A similar calculation shows that this is also the value of G_c. All choices of indices lead to the same value for the integral G.

3. Some multiloop calculations

This section uses RMT for the evaluation of two multidimensional integrals of the form

$$I = \int_0^{\infty} x_1^{\nu_1-1} \ldots \int_0^{\infty} x_N^{\nu_N-1} \, f(x_1,\ldots,x_N)\, dx_1 \cdots dx_N. \tag{3.1}$$

As in the one-dimensional case, the function f is assumed to admit a Taylor expansion

$$\begin{aligned} f(x_1,\ldots,x_N) = \sum_{l_1=0}^{\infty} \ldots \sum_{l_N=0}^{\infty} \frac{(-1)^{l_1}}{l_1!} \ldots \frac{(-1)^{l_N}}{l_N!} \; \varphi(l_1,\ldots,l_N) \\ \times\; x_1^{a_{11}l_1+\ldots+a_{1N}l_N+b_1} \ldots x_N^{a_{N1}l_1+\ldots+a_{NN}l_N+b_N}, \end{aligned}$$

so that I is expressed as

$$\begin{aligned} I = \int_0^{\infty} \cdots \int_0^{\infty} \sum_{l_1=0}^{\infty} \ldots \sum_{l_N=0}^{\infty} \frac{(-1)^{l_1}}{l_1!} \ldots \frac{(-1)^{l_N}}{l_N!} \; \varphi(l_1,..,l_N) \\ \times\; x_1^{a_{11}l_1+\ldots+a_{1N}l_N+\widetilde{b}_1} .. \, x_N^{a_{N1}l_1+\ldots+a_{NN}l_N+\widetilde{b}_N} \frac{dx_1}{x_1} \cdots \frac{dx_N}{x_N} \end{aligned}$$

with $\widetilde{b}_i = \nu_i + b_i$ $(i = 1,\ldots,N)$.

Applying RMT in iterative form gives

$$I = \frac{1}{|\det(\mathbf{A})|} \, \Gamma(-l_1^*) \ldots \Gamma(-l_N^*) \; \varphi(l_1^*,\ldots,l_N^*) \tag{3.2}$$

where $\mathbf{A} = (a_{ij})$ and $\mathbf{l}^* = (l_1^*,\cdots,l_N^*)$ is the solution of the linear system $\mathbf{A}\mathbf{l}^* = -\widetilde{\mathbf{b}}$. Details of the proof of this result appear in [1].

Example 3.1. Massive sunset diagram. The first example is associated to the diagram shown in Figure 1. In the momentum space, the integral is given by

$$G := \int \frac{1}{[q^2 - M^2]^{a_1}} \frac{1}{\left[(q_1 - q_2)^2\right]^{a_2}} \frac{1}{[(p+q_2)^2]^{a_3}} \frac{d^D q_1}{i\pi^{D/2}} \frac{d^D q_2}{i\pi^{D/2}} \tag{3.3}$$

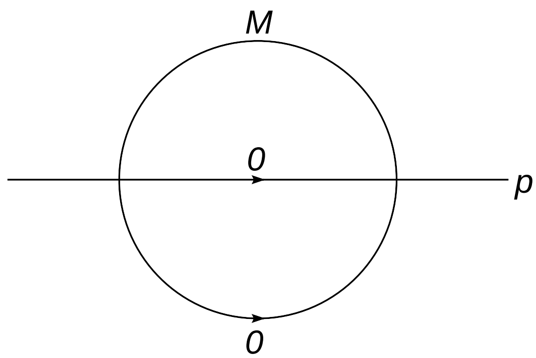

FIGURE 1. The sunset diagram

In terms of the Schwinger parametrization, G becomes

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp\left(x_1 M^2\right) \exp\left(-\frac{x_1 x_2 x_3}{x_1x_2 + x_1x_3 + x_2x_3}\, p^2\right)}{(x_1x_2 + x_1x_3 + x_2x_3)^{\frac{D}{2}}} d\overrightarrow{x},$$

where $d\overrightarrow{x} = x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} dx_1\, dx_2\, dx_3$.

The evaluation is described here only[2] in special case $p^2 = M^2$. The general case is only algebraically more complicated. The integral reduces to

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp\left[\frac{x_1^2 (x_2 + x_3)}{x_1(x_2+x_3) + x_2x_3}\, M^2\right]}{[x_1(x_2+x_3) + x_2x_3]^{\frac{D}{2}}} d\overrightarrow{x}.$$

The expansion of the exponential function yields

$$\exp\left(\frac{x_1^2(x_2+x_3)}{x_1(x_2+x_3) + x_2x_3}\, M^2\right) = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1}(x_2+x_3)^{n_1}}{[x_1(x_2+x_3) + x_2x_3]^{n_1}}$$

so that

(3.4)

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1}(x_2+x_3)^{n_1}}{[x_1(x_2+x_3) + x_2x_3]^{\frac{D}{2}+n_1}} d\overrightarrow{x}.$$

The binomial theorem

$$(x+y)^a = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(-a+n)}{\Gamma(-a)}\, x^{a-n} y^n \tag{3.5}$$

[2]This special case is of phyiscal interest.

gives

$$\frac{1}{\left[x_1\left(x_2+x_3\right)+x_2x_3\right]^{\frac{D}{2}+n_1}} = \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!} \frac{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}{\Gamma\left(\frac{D}{2}+n_1\right)} x_1^{-\frac{D}{2}-n_1-n_2} \left(x_2+x_3\right)^{-\frac{D}{2}-n_1-n_2} x_2^{n_2} x_3^{n_2},$$

and (3.4) becomes

$$\begin{aligned} G &= \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(-1)^{n_1}}{n_1!}\frac{(-1)^{n_2}}{n_2!}\left(M^2\right)^{n_1} \\ &\quad \times \frac{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}{\Gamma\left(\frac{D}{2}+n_1\right)} x_1^{n_1-\frac{D}{2}-n_2} x_2^{n_2} x_3^{n_2} \left(x_2+x_3\right)^{-\frac{D}{2}-n_2} d\overrightarrow{x}. \end{aligned}$$

The final expansion

$$(x_2+x_3)^{-\frac{D}{2}-n_2} = \sum_{n_3=0}^{\infty} \frac{(-1)^{n_3}}{n_3!} \frac{\Gamma\left(\frac{D}{2}+n_2+n_3\right)}{\Gamma\left(\frac{D}{2}+n_2\right)} x_2^{-\frac{D}{2}-n_2-n_3} x_3^{n_3} \tag{3.6}$$

expresses the integral in the form required for the application of RMT. This gives

$$\begin{aligned} G &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \\ &\quad \times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{(-1)^{n_1}}{n_1!}\frac{(-1)^{n_2}}{n_2!}\frac{(-1)^{n_3}}{n_3!} x_1^{a_1-\frac{D}{2}+n_1-n_2} x_2^{a_2-\frac{D}{2}-n_3} x_3^{a_3+n_2+n_3} \\ &\quad \times \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \frac{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}{\Gamma\left(\frac{D}{2}+n_1\right)} \frac{\Gamma\left(\frac{D}{2}+n_2+n_3\right)}{\Gamma\left(\frac{D}{2}+n_2\right)} \left(-M^2\right)^{n_1} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}. \end{aligned} \tag{3.7}$$

Therefore

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \Gamma(n_1^*)\Gamma(n_2^*)\Gamma(n_3^*) \frac{\Gamma\left(\frac{D}{2}-n_1^*-n_2^*\right)}{\Gamma\left(\frac{D}{2}-n_1^*\right)} \frac{\Gamma\left(\frac{D}{2}-n_2^*-n_3^*\right)}{\Gamma\left(\frac{D}{2}-n_2^*\right)} \left(-M^2\right)^{-n_1^*}, \tag{3.8}$$

where the indices $\{n_i^*\}$ are given by the unique solution to the linear system

$$\begin{aligned} n_1-n_2 &= a_1-D/2, \\ n_3 &= -a_2+D/2, \\ n_2+n_3 &= a_3, \end{aligned}$$

associated to (3.7). The solutions are

$$n_1^* = a_1+a_2+a_3-D,\ n_2^* = a_2+a_3-D/2,\ n_3^* = D/2-a_2. \tag{3.9}$$

The value of the integral G is finally given by

$$\begin{aligned} G &= (-1)^{-D} \frac{\Gamma(a_1+a_2+a_3-D)\Gamma\left(a_2+a_3-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\ &\quad \times \frac{\Gamma\left(\frac{D}{2}-a_3\right)\Gamma(2D-a_1-2a_2-2a_3)}{\Gamma\left(\frac{3D}{2}-a_1-a_2-a_3\right)\Gamma(D-a_2-a_3)} \left(-M^2\right)^{D-a_1-a_2-a_3}. \end{aligned}$$

Example 3.2. Massless three loops ladder diagram. The last example evaluates the integral[3] associated to the diagram seen in figure 2

$$G=\frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)...\Gamma(a_{10})}\int_0^\infty ... \int_0^\infty \frac{\exp\left(-\dfrac{x_1x_4x_7x_{10}}{U}\,t\right)}{U^{\frac{D}{2}}}d\overrightarrow{x}. \tag{3.10}$$

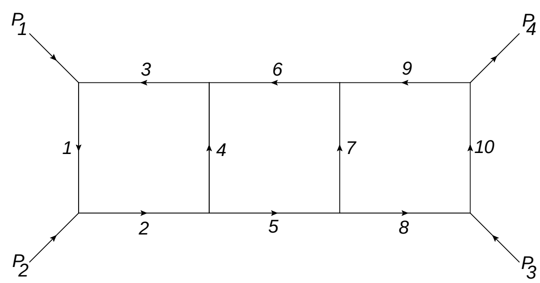

FIGURE 2. The 3-loop ladder

Here $d\overrightarrow{x}=\prod_{j=1}^{10}dx_j\,x_j^{a_j-1}$ and U is a polynomial given by

$$U=x_5\left(x_7+\mathbf{f}_1\right)\left(\mathbf{f}_2+x_4\right)+x_6\left(x_7+\mathbf{f}_1\right)\left(\mathbf{f}_2+x_4\right)+x_4\left(x_7+\mathbf{f}_1\right)\mathbf{f}_2+x_7\left(\mathbf{f}_2+x_4\right)\mathbf{f}_1,$$

with

$$\mathbf{f}_1=x_8+x_9+x_{10} \text{ and } \mathbf{f}_2=x_1+x_2+x_3. \tag{3.11}$$

Expanding the exponential term yields

$$G=\frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})}\int_0^\infty\cdots\int_0^\infty\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}t^{n_1}\frac{x_1^{n_1}x_4^{n_1}x_7^{n_1}x_{10}^{n_1}}{U^{\frac{D}{2}+n_1}}d\overrightarrow{x}, \tag{3.12}$$

and expanding U by the multinomial theorem

$$\begin{aligned}(x_1+...+x_{k-1}+x_k)^a &= \sum_{n_1=0}^{\infty}...\sum_{n_{k-1}=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}...\frac{(-1)^{n_{k-1}}}{n_{k-1}!}\frac{\Gamma(-a+n_1+...+n_{k-1})}{\Gamma(-a)}\\ &\quad\times x_1^{n_1}...x_{k-1}^{n_{k-1}}x_k^{a-n_1-...-n_{k-1}}\end{aligned}$$

gives

$$\begin{aligned}U^{-\frac{D}{2}-n_1} &= \sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\frac{(-1)^{n_2}}{n_2!}\frac{(-1)^{n_3}}{n_3!}\frac{(-1)^{n_4}}{n_4!}\frac{\Gamma\left(\frac{D}{2}+n_1+n_2+n_3+n_4\right)}{\Gamma\left(\frac{D}{2}+n_1\right)}\\ &\quad\times\mathbf{f}_1^{-\frac{D}{2}-n_1-n_2-n_3-n_4}\mathbf{f}_2^{n_2}\left(x_7+\mathbf{f}_1\right)^{n_2+n_3+n_4}\left(x_4+\mathbf{f}_2\right)^{-\frac{D}{2}-n_1-n_2}\\ &\quad\times x_4^{n_2}x_5^{n_3}x_6^{n_4}x_7^{-\frac{D}{2}-n_1-n_2-n_3-n_4}.\end{aligned}$$

[3]In a simplified physical situation, where the conditions $P_i^2=0$ for $1\le i\le 4$ and $s=0$ are imposed

Similarly,

$$(x_7+\mathbf{f}_1)^{n_2+n_3+n_4} = \sum_{n_5=0}^{\infty}\frac{(-1)^{n_5}}{n_5!}\frac{\Gamma\left(-n_2-n_3-n_4+n_5\right)}{\Gamma\left(-n_2-n_3-n_4\right)}x_7^{n_5}\mathbf{f}_1^{n_2+n_3+n_4-n_5},$$

$$(x_4+\mathbf{f}_2)^{-\frac{D}{2}-n_1-n_2} = \sum_{n_6=0}^{\infty}\frac{(-1)^{n_6}}{n_6!}\frac{\Gamma\left(\frac{D}{2}+n_1+n_2+n_6\right)}{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}x_4^{n_6}\mathbf{f}_2^{-\frac{D}{2}-n_1-n_2-n_6},$$

and

$$\begin{aligned}\mathbf{f}_1^{-\frac{D}{2}-n_1-n_5} &= \sum_{n_7=0}^{\infty}\sum_{n_8=0}^{\infty}\frac{(-1)^{n_7}}{n_7!}\frac{(-1)^{n_8}}{n_9!}\frac{\Gamma\left(\frac{D}{2}+n_1+n_5+n_7+n_8\right)}{\Gamma\left(\frac{D}{2}+n_1+n_5\right)}\\ &\quad\times x_8^{n_7}x_9^{n_8}x_{10}^{-\frac{D}{2}-n_1-n_5-n_7-n_8},\end{aligned}$$

$$\begin{aligned}\mathbf{f}_2^{-\frac{D}{2}-n_1-n_6} &= \sum_{n_9=0}^{\infty}\sum_{n_{10}=0}^{\infty}\frac{(-1)^{n_9}}{n_9!}\frac{(-1)^{n_{10}}}{n_{10}!}\frac{\Gamma\left(\frac{D}{2}+n_1+n_6+n_9+n_{10}\right)}{\Gamma\left(\frac{D}{2}+n_1+n_6\right)}\\ &\quad\times x_1^{n_9}x_2^{n_{10}}x_3^{-\frac{D}{2}-n_1-n_6-n_9-n_{10}},\end{aligned}$$

finally produce

$$\begin{aligned}G &= \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})}\int_0^{\infty}\cdots\int_0^{\infty}\sum_{n_1=0}^{\infty}...\sum_{n_{10}=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}...\frac{(-1)^{n_{10}}}{n_{10}!}\,\varphi(n_1,...,n_{10})\\ &\quad\times x_1^{a_1+n_1+n_9}x_2^{a_2+n_{10}}x_3^{a_3-\frac{D}{2}-n_1-n_6-n_9-n_{10}}x_4^{a_4+n_1+n_2+n_6}x_5^{a_5+n_3}x_6^{a_6+n_4}\\ &\quad\times x_7^{a_7-\frac{D}{2}-n_2-n_3-n_4+n_5}x_8^{a_8+n_7}x_9^{a_9+n_8}x_{10}^{a_{10}-\frac{D}{2}-n_5-n_7-n_8},\ \frac{dx_1}{x_1}\cdots\frac{dx_{10}}{x_{10}}\end{aligned}$$

with the notation

$$\begin{aligned}\varphi(n_1,...,n_{10}) = &\frac{\Gamma\left(\frac{D}{2}+n_1+n_2+n_3+n_4\right)}{\Gamma\left(\frac{D}{2}+n_1\right)}\frac{\Gamma\left(-n_2-n_3-n_4+n_5\right)}{\Gamma\left(-n_2-n_3-n_4\right)}\frac{\Gamma\left(\frac{D}{2}+n_1+n_2+n_6\right)}{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}\\ &\times\frac{\Gamma\left(\frac{D}{2}+n_1+n_5+n_7+n_8\right)}{\Gamma\left(\frac{D}{2}+n_1+n_5\right)}\frac{\Gamma\left(\frac{D}{2}+n_1+n_6+n_9+n_{10}\right)}{\Gamma\left(\frac{D}{2}+n_1+n_6\right)}t^{n_1}.\end{aligned}$$

A direct application of RMT gives

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})}\frac{1}{|\det(\mathbf{A})|}\,\varphi\left(-n_1^*,...,-n_{10}^*\right)\prod_{j=1}^{10}\Gamma\left(n_j^*\right) \tag{3.13}$$

where $\{n_i^*\}$ are solutions to the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1^* \\ n_2^* \\ n_3^* \\ n_4^* \\ n_5^* \\ n_6^* \\ n_7^* \\ n_8^* \\ n_9^* \\ n_{10}^* \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 - \frac{D}{2} \\ a_4 \\ a_5 \\ a_6 \\ a_7 - \frac{D}{2} \\ a_8 \\ a_9 \\ a_{10} - \frac{D}{2} \end{pmatrix}$$

given by

$$\begin{aligned} n_1^* &= -\tfrac{3D}{2} + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}, \\ n_2^* &= D - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}, \\ n_3^* &= a_5, \\ n_4^* &= a_6, \\ n_5^* &= \tfrac{D}{2} - a_8 - a_9 - a_{10}, \\ n_6^* &= \tfrac{D}{2} - a_1 - a_2 - a_3, \\ n_7^* &= a_8, \\ n_8^* &= a_9, \\ n_9^* &= \tfrac{3D}{2} - a_2 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}, \\ n_{10}^* &= a_2. \end{aligned} \tag{3.14}$$

This gives the value of the diagram as

$$\begin{aligned} G = \quad & (-1)^{-\frac{3D}{2}} \frac{\Gamma\left(\frac{D}{2} - a_{89,10}\right)\Gamma\left(\frac{D}{2} - a_{123}\right)\Gamma\left(\frac{3D}{2} - a_{23456789,10}\right)\Gamma\left(\frac{3D}{2} - a_{123456789}\right)}{\Gamma(a_1)\Gamma(a_4)\Gamma(a_7)\Gamma(a_{10})\Gamma(2D - a_{123456789,10})} \\ & \times \frac{\Gamma\left(a_{123456789,10} - \frac{3D}{2}\right)\Gamma(D - a_{56789,10})\Gamma(D - a_{123456})\Gamma\left(\frac{D}{2} - a_7\right)\Gamma\left(\frac{D}{2} - a_4\right)}{\Gamma(D - a_{789,10})\Gamma(D - a_{1234})\Gamma\left(\frac{3D}{2} - a_{1234567}\right)\Gamma\left(\frac{3D}{2} - a_{456789,10}\right)} \\ & \times t^{\frac{3D}{2} - a_{123456789,10}}, \end{aligned}$$

with the notation

$$a_{ijk\ldots} = a_i + a_j + a_k + \ldots. \tag{3.15}$$

An important special case, when all powers a_i of propagators are 1, is

$$G = (-1)^{-\frac{3D}{2}} \frac{\Gamma\left(10 - \frac{3D}{2}\right)\Gamma\left(\frac{D}{2} - 3\right)^2\Gamma\left(\frac{3D}{2} - 9\right)^2\Gamma(D-6)^2\Gamma\left(\frac{D}{2} - 1\right)^2}{\Gamma(2D - 10)\Gamma(D-4)^2\Gamma\left(\frac{3D}{2} - 7\right)^2} t^{\frac{3D}{2} - 10}.$$

4. Conclusions

This paper presents a technique for the evaluation of a large variety of intergrals coming from Feynman diagrams. The advantage over previous methods is that the evaluation is reduced to series expansions of the integrand coupled with the solution of a linear system of equations.

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