THE CAUCHY-SCHLÖMILCH TRANSFORMATION

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Abstract. The Cauchy-Schlömilch transformation states that for a function f and a, $b > 0$, the integral of $f(x^2)$ and $af((ax-bx^{-1})^2$ over the interval $[0, \infty)$ are the same. This elementary result is used to evaluate many non-elementary definite integrals, most of which cannot be obtained by symbolic packages. Applications to probability distributions is also given.

1. Introduction

The problem of analytic evaluations of definite integrals has been of interest to scientists for a long time. The central question can be stated as follows:

given a class of functions $\mathfrak{F}$ *and an interval* $[a,b] \subset \mathbb{R}$, *express the integral of* $f \in \mathfrak{F}$

$$I = \int_a^b f(x)\, dx,$$

in terms of special values of functions in an enlarged class $\mathfrak{G}$.

Many methods for the evaluation of definite integrals have been developed since the early stages of Integral Calculus, which resulted in a variety of ad-hoc techniques for producing closed-form expressions. Although a general procedure applicable to all integrals is undoubtedly unattainable, it is within reason to expect a systematic cataloguing procedure for large groups of definite integrals. To this effect, one of the authors has instituted a project to verify all the entries in the popular table by I. S. Gradshteyn and I. M. Ryzhik [13]. The website

`http://www.math.tulane.edu/~vhm/web_html/pap-index.html`

contains a series of papers as a treatment to the above-alluded project.

Naturally, any document containing a large number of entries, such as the table [13] or the encyclopedic treatise [20], is likely to contain errors, many of which arising from transcription from other tables. The earliest extensive table of integrals still accessible is [2], compiled by Bierens de Haan who also presented in [3] a survey of the methods employed in the verification of the entries from [13]. These tables form the main source for [13].

The revision of integral tables is nothing new. C. F. Lindman [17] compiled a long list of errors from the table by Bierens de Haan [4]. The editors of [13] maintain the webpage

`http://www.mathtable.com/gr/`

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where the corrections to the table are stored.

Many techniques have been developed for evaluating definite integrals, and the goal of this paper is to present one such method popularized by Schlömilch in [24]. The identity (2.1) below appeared in [6], where it was called the *Schlömilch transform*, although it was used by J. Liouville [18] to evaluate the integral

$$\int_0^1 \frac{t^{\mu+1/2}(1-t)^{\mu-1/2}\,dt}{(a+bt-ct^2)^{\mu+1}}, \tag{1.1}$$

and in [19] Liouville quotes a letter from Schlömilch in which he describes his approach to (1.1) via the formula

$$\int_0^\infty F\left(\frac{\alpha}{x}+\gamma x\right)\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{\gamma}}\int_0^\infty F(2\sqrt{\alpha\gamma}+y)\frac{dy}{\sqrt{y}} \tag{1.2}$$

to derive the reduction

$$\int_0^\infty \frac{x^{m+1/2}\,dx}{(\alpha+\beta x+\gamma x^2)^{m+1}} = \frac{1}{\sqrt{\gamma}}\int_0^\infty \frac{y^{1/2-m}\,dy}{(\beta+2\sqrt{\alpha\gamma}+y)^{m+1}}. \tag{1.3}$$

The integral (1.1) is then evaluated in terms of the *beta function*. Schlömilch also states that his method can be found in a note by A. Cauchy [8] published some 35 years earlier[1]. In view of this historical precedence, the name *Cauchy-Schlömilch* adopted here seems to be more justified. Some illustrative examples appear in the text [21].

We present here a variety of definite integrals evaluated by use of the Schlömilch transform and its extensions. Several of the examples are not computable by the current symbolic languages. For each evaluation presented in the upcoming sections, we considered its computation using `Mathematica`. Naturally, the statement 'the integral can not be computed symbolically' has to be complemented with the phrase 'at the present date'.

2. The Cauchy-Schlömilch transformation

In this section we present the basic result accompanied with initial examples. Further applications are discussed in the remaining sections.

Theorem 2.1. [Cauchy-Schlömilch] Let a, $b>0$ and assume that f is a continuous function for which the integrals in (2.1) are convergent. Then

$$\int_0^\infty f\left(\left(ax-bx^{-1}\right)^2\right)\,dx \quad = \quad \frac{1}{a}\int_0^\infty f(y^2)\,dy. \tag{2.1}$$

Proof. The change of variables $t=b/ax$ yields

$$\begin{aligned} I &= \int_0^\infty f\left((ax-b/x)^2\right)\,dx \\ &= \frac{b}{a}\int_0^\infty f\left((at-b/t)^2\right)\,t^{-2}\,dt. \end{aligned}$$

The average of these two representations, followed by the change of variables $u = ax-b/x$ completes the proof. □

The next result is a direct consequence of Theorem 2.1.

[1]This note is available at `http://gallica.bnf.fr/ark:/12148/bpt6k90193x.zoom.f526`

Corollary 2.2. Preserve the assumptions of Theorem 2.1. Let $h_n = \sum_{k=0}^{n} c_k x^{2k+1}$ be an odd polynomial. Define

$$g_n(x) = x\left(\sum_{k=0}^{n} d_k x^k\right)^2, \tag{2.2}$$

where

$$d_k = \sum_{j=k}^{n} \binom{k+j}{2k} \frac{2j+1}{2k+1} (ab)^{j-k} c_j. \tag{2.3}$$

Then

$$\int_0^\infty f\left(\left[h_n(ax) - h_n(bx^{-1})\right]^2\right) dx = \frac{1}{a}\int_0^\infty f(g_n(y^2))\, dy. \tag{2.4}$$

Proof. Denote

$$H_n(x) = h_n(ax) - h_n(bx^{-1}) = \sum_{k=0}^{n} c_k \psi_k(x) \tag{2.5}$$

where $\psi_k(x) = (ax)^{2k+1} - (bx^{-1})^{2k+1}$ and let $\phi_k(x) = (ax - bx^{-1})^{2k+1}$. Then, the polynomials ψ_k and ϕ_k obey the transformation rule

$$\psi_k(x) = \sum_{j=0}^{k} \binom{k+j}{2j} \frac{2k+1}{2j+1} (ab)^{k-j} \phi_j(x). \tag{2.6}$$

The expression for H_n in terms of ϕ_k follows directly from this. Moreover, $H_n^2(x) = g_n\left((ax - bx^{-1})^2\right)$ and the result is obtained by applying the Cauchy-Schlömilch formula to $f(g_n)$. □

Example 2.3. For $n \in \mathbb{N}$,

$$\int_0^\infty \left[(x^2 + x^{-6})(x^4 - x^2 + 1) - 1\right] e^{-(x^7 - x^{-7})^{2n}}\, dx = \frac{1}{14n}\Gamma\left(\frac{1}{2n}\right).$$

In order to verify this value, write the Laurent polynomials of the integrand according to (2.3)

$$x^7 - x^{-7} = 7y + 14y^3 + 7y^5 + y^7,$$

and write the integrand as a function of $y = x - x^{-1}$, to obtain

$$7(x^2 + x^{-6})(x^4 - x^2 + 1) - 7 = 7 + 42y^2 + 35y^4 + 7y^6.$$

Implement (2.4) on

$$f(y^2) = (7 + 42y^2 + 35y^4 + 7y^6)e^{-(y^2(7+14y^2+7y^4+y^6)^2)^n}, \tag{2.7}$$

to yield

$$\int_0^\infty 7\left[(x^2 + x^{-6})(x^4 - x^2 + 1) - 1\right] e^{-(x^7 - x^{-7})^{2n}}\, dx = \int_0^\infty f(y^2) dy.$$

The last step is an outcome of the substitution $z = 7y + 14y^3 + 7y^5 + y^7$. Hence

$$\begin{aligned}\int_0^\infty f(y^2) dy &= \int_0^\infty (7 + 42y^2 + 35y^4 + 7y^6)e^{-(7y+14y^3+7y^5+y^7)^{2n}}\, dy \\ &= \int_0^\infty e^{-z^{2n}}\, dz = \frac{1}{2n}\Gamma\left(\frac{1}{2n}\right).\end{aligned}$$

Example 2.4. Proceeding as in the previous example, we obtain

$$\int_0^\infty \left[7(x^2+x^{-6})(x^4-x^2+1)-6\right] e^{-(x+x^7-x^{-1}-x^{-7})^{2n}} \, dx = \frac{1}{2n}\Gamma\left(\frac{1}{2n}\right).$$

The details are left to the reader.

Example 2.5. The choice $y = x - x^{-1}$, $x^3 - x^{-3} = 3y + y^3$ followed by $z = 3y + y^3$ produces

$$\begin{aligned}
\int_0^\infty \frac{x^4-x^2+1}{x^2} \prod_{j=0}^\infty \left[1+(x^3-x^{-3})^2\,\nu^{2j}\right]^{-1} \, dx = \\
&= \int_0^\infty (1+y^2) \prod_{j=0}^\infty \left[1+(3y+y^3))^2\,\nu^{2j}\right]^{-1} \, dy \\
&= \frac{1}{3}\int_0^\infty \prod_{j=0}^\infty (1+z^2\nu^{2j})^{-1} \, dz \\
&= \frac{\pi}{6}(1+\nu+\nu^3+\nu^6+\nu^{10}+\cdots)^{-1}.
\end{aligned}$$

3. An integral due to Laplace

The example described in this section is the original problem to which the Cauchy-Schlömilch transformation was applied.

Example 3.1. The *normal integral* is

$$\int_0^\infty e^{-y^2} \, dy = \frac{\sqrt{\pi}}{2}. \tag{3.1}$$

The reader will find in [6] a variety of proofs of this fundamental identity.

Take $f(x) = e^{-x}$ in Theorem 2.1 to obtain

$$\int_0^\infty e^{-(ax-b/x)^2} \, dx = \frac{\sqrt{\pi}}{2a}. \tag{3.2}$$

Expanding the integrand and replacing the parameters a and b by their square roots produces entry 3.325 in [13]:

$$\int_0^\infty \exp\left(-ax^2-b/x^2\right) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}. \tag{3.3}$$

The change of variable $x = \sqrt{b}t/\sqrt{a}$ shows that the result (3.3) can be written in terms of a single parameter $c = ab$ as

$$\int_0^\infty e^{-c(t-1/t)^2} \, dt = \frac{1}{2}\sqrt{\frac{\pi}{c}}. \tag{3.4}$$

Example 3.2. A host of other entries in [13] are amenable to the Cauchy-Schlömilch transformation. For example, 3.324.2 states that

$$\int_{-\infty}^\infty \exp\left[-(x-b/x)^{2n}\right] \, dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right). \tag{3.5}$$

This is now evaluated by choosing $f(x) = e^{-x^n}$ in Theorem 2.1 so that

$$\int_{-\infty}^\infty \exp\left[-(x-b/x)^{2n}\right] \, dx = 2\int_0^\infty e^{-y^{2n}} \, dy.$$

The change of variables $t = y^{2n}$ and the integral representation for the *gamma function*

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1}\, dt \tag{3.6}$$

imply (3.5).

Example 3.3. The expression $t - 1/t$ in (3.4) suggests a natural change of variables $t = e^u$. This yields

$$\int_{-\infty}^{\infty} e^{u - c\,\sinh^2 u}\, du = \sqrt{\frac{\pi}{c}}. \tag{3.7}$$

The latest version of `Mathematica` is unable to produce this result when c is an arbitrary parameter. It does evaluate (3.7) if c is assigned a specific real value.

4. An integral with three parameters

The introduction of parameters in a definite integral provides a greater flexibility in its evaluation. Many classical integrals are presented in [5] as special cases of the next theorem, which now appears as 3.242.2 in [13]. The proof given below is in the spirit of the original observation of A. Cauchy.

Theorem 4.1. Let

$$\begin{aligned}
I_1 &= \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^c \cdot \frac{x^2+1}{x^b+1} \frac{dx}{x^2} \\
I_2 &= \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^c \frac{dx}{x^2} \\
I_3 &= \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^c dx \\
I_4 &= \frac{1}{2} \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^c \frac{x^2+1}{x^2}\, dx.
\end{aligned}$$

Then $I_1 = I_2 = I_3 = I_4$ and this common value is

$$I(a,b;c) = 2^{-1/2-c}(1+a)^{1/2-c} B\left(c - \tfrac{1}{2}, \tfrac{1}{2}\right). \tag{4.1}$$

Proof. Observe that if g satisfies $g(1/x) = x^2 g(x)$, differentiation with respect to the parameter b shows that the integral of $g(x)/(x^b+1)$ over $[0,\infty)$ is independent of b. This proves the equivalence of the four stated integrals.

Theorem 2.1 is now used to evaluate I_3. For any function f, the Cauchy-Schlömilch transformation gives

$$\begin{aligned}
\int_0^\infty f\left(\frac{x^2}{x^4 + 2ax^2 + 1} \right) dx &= \int_0^\infty f\left(\frac{1}{(x - x^{-1})^2 + 2a + 2} \right) dx \\
&= \int_0^\infty f\left(\frac{1}{x^2 + 2(a+1)} \right).
\end{aligned}$$

Apply this to $f(x) = x^c$ and use the change of variables $u = \frac{2(a+1)}{x^2+2(a+1)}$ to produce

$$\int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^c dx = \frac{1}{2} \left[2(a+1) \right]^{\frac{1}{2}-c} \int_0^1 u^{c-3/2}(1-u)^{-1/2}.$$

This last integral is the special value $B(c - \frac{1}{2}, \frac{1}{2})$ of Euler's beta function. □

The next theorem presents an alternative form of the integral in Theorem 2.1.

Theorem 4.2. For any function f,

$$\int_0^\infty f\left(\frac{bx^2}{x^4 + 2ax^2 + 1}\right) dx = \frac{\sqrt{b}}{2\sqrt{a_*}} \int_0^1 \frac{f(a_* t)}{\sqrt{t(1-t)}} \frac{dt}{t}, \tag{4.2}$$

where $a_* = \frac{b}{2(1+a)}$.

Proof. This follows from the identity in Theorem 2.1 and the change of variable $t = 2(a+1)/[x^2 + 2(a+1)]$. □

The *master formula* (4.1) yields many other evaluations of definite integrals; see [7] for some of them. The next theorem provides a new class of integrals that are derived from (4.1).

Theorem 4.3. Suppose

$$f(x) = \sum_{n=1}^{\infty} c_n x^n$$

be an analytic function with $f(0) = 0$. Then

$$\int_0^\infty f\left(\frac{x^2}{x^4 + 2ax^2 + 1}\right) dx = \frac{\pi}{2^{3/2}\sqrt{1+a}} \sum_{n=0}^{\infty} c_{n+1} \binom{2n}{n} u^n, \tag{4.3}$$

where $u = \frac{1}{8(1+a)}$.

Proof. Integrate term-by-term and use the value

$$B\left(m + \tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\pi}{2^{2m}} \binom{2m}{m}$$

to simplify the result. □

5. Exponentials and Bessel functions

This section describes the application of Theorem 4.3 to a number of definite integrals. The Taylor expansion of $f(x) = 1 - e^{-bx}$ with $b > 0$ leads to an integral that can be evaluated in terms of the *modified Bessel* functions $I_\nu(x)$ defined by the series

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{x^{\nu+2j}}{j!\Gamma(\nu+j+1)\, 2^{\nu+2j}}. \tag{5.1}$$

In particular

$$I_0(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j} j!^2} \text{ and } I_1(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2^{2j+1} j!(j+1)!}. \tag{5.2}$$

Although, as will be seen below, our results below have more direct derivations, the following procedure is more informative.

Example 5.1. For $a > -1$ and $b > 0$, let $c = \frac{b}{8(1+a)}$. Then

$$\int_0^\infty \left(1 - e^{-bx^2/(x^4+2ax^2+1)}\right) dx = \frac{\pi b e^{-2c}}{2^{3/2}\sqrt{1+a}} \left[I_0(2c) + I_1(2c) \right].$$

The function $f(x) = 1 - e^{-bx}$ has coefficients $c_n = (-1)^{n+1} b^n/n!$ and (4.3) yields

$$\int_0^\infty \left(1 - e^{-bx^2/(x^4+2ax^2+1)}\right) dx = \frac{\pi b}{2^{3/2}\sqrt{1+a}} h(-bu)$$

where $u = 1/8(1+a)$ and

$$h(x) = \sum_{n=0}^\infty \frac{\binom{2n}{n}}{(n+1)!} x^n. \tag{5.3}$$

The result now follows from the relation $c = bu$ and an identification of the series h in terms of Bessel functions.

Proposition 5.2. The following identity holds:

$$\sum_{n=0}^\infty \frac{\binom{2n}{n}}{(n+1)!} x^n = e^{2x} \left[I_0(2x) - I_1(2x) \right].$$

We present two different proofs. The first one is elementary and is based on the WZ-method described in [23]. `Mathematica` actually provides a third proof by direct evaluation of the series.

Proof. The expansion (5.2) yields

$$I_0(2x) - I_1(2x) = \sum_{r=0}^\infty \frac{x^r}{b_r}, \tag{5.4}$$

where

$$b_r = \begin{cases} j!^2 & \text{if } r = 2j \\ -j!(j+1)! & \text{if } r = 2j+1. \end{cases}$$

Multiplying (5.4) with the series for e^{2x} lends itself to an equivalnet formulation of the claim as the identity

$$\sum_{j=0}^k \frac{(-1)^j}{2^j} \binom{k}{j} \binom{j}{\lfloor j/2 \rfloor} = \frac{1}{2^k(k+1)} \binom{2k}{k}. \tag{5.5}$$

The upper index of the sum is extended to infinity and (5.5) is written as

$$\sum_{j\geq 0} \frac{(-1)^j}{2^j} \binom{k}{j} \binom{j}{\lfloor j/2 \rfloor} = \frac{1}{2^k(k+1)} \binom{2k}{k}.$$

The even and odd indices are considered separately. Define

$$S_e := \sum_{j \geq 0} \frac{1}{2^{2j}} \binom{k}{2j} \binom{2j}{j} \text{ and } S_o := \sum_{j\geq 0} \frac{1}{2^{2j+1}} \binom{k}{2j+1} \binom{2j+1}{j}.$$

The result is obtained from the values

$$S_e = \frac{1}{2^k} \binom{2k}{k} \text{ and } S_o = \frac{k}{(k+1)\,2^k} \binom{2k}{k}. \tag{5.6}$$

To establish (5.6), the WZ-method is applied to the functions

$$S_e^*(k) = S_e \binom{2n}{k}^{-1} 2^k \text{ and } S_o^*(k) = S_o \binom{2k}{k}^{-1} \frac{k+1}{k}.$$

The output is that both S_e^* and S_o^* satisfy the recurrence $a_{k+1} - a_k = 0$ with *certificates*

$$\frac{-4j^2}{(2k+1)(k+1+2j)} \text{ and } \frac{-4j(j+1)}{(2k+1)(k-2j)},$$

respectively. The initial conditions $S_e^*(0) = S_o^*(0) = 1$ give $S_e^*(k) \equiv S_o^*(k) \equiv 1$. □

As promised above we present an alternative proof of 5.2 based on Theorem 4.2.

Proof. Theorem 4.2 gives

$$I := \int_0^\infty \left(1 - e^{-bx^2/(x^4+2ax^2+1)}\right) dx = \frac{\sqrt{b}}{2\sqrt{a^*}} \int_0^1 \frac{1-e^{-a^*t}}{t} \frac{dt}{\sqrt{t(1-t)}}.$$

The latter is known as *Frullani integral* and can be written as

$$I = \frac{\sqrt{ba^*}}{2} \int_0^1 \int_0^1 \frac{e^{-a^*ty}}{\sqrt{t(1-t)}} dy\, dt. \tag{5.7}$$

Exchanging the order of integration, the inner integral is a well-known Laplace transform [11](p. 366, 19.5.11 with $n = 0$)

$$\int_0^1 \frac{e^{-\omega t}\, dt}{\sqrt{t(1-t)}} = \pi e^{-\omega/2} I_0\left(\tfrac{\omega}{2}\right) \tag{5.8}$$

whence we find

$$I = \frac{\pi\sqrt{b}}{\sqrt{a^*}} \int_0^{a^*/2} e^{-t} I_0(t)\, dt.$$

The relation

$$\frac{d}{dt}\left(te^{-t}(I_0(t) + I_1(t))\right) = e^{-t} I_0(t) \tag{5.9}$$

now completes the proof. □

Example 5.3. Choosing $a = 0$ and $b = 4$ gives

$$\int_0^\infty \left(1 - e^{-4x^2/(x^4+1)}\right) dx = \frac{\pi\sqrt{2}}{e} \left(I_0(1) + I_1(1)\right).$$

Example 5.4. The values $a = 1$ and $b = 8$ yield

$$\int_0^\infty \left(1 - e^{-8x^2/(x^2+1)^2}\right) dx = \frac{2\pi}{e} \left(I_0(1) + I_1(1)\right).$$

`Mathematica` is unable to evaluate these two examples.

6. Trigonometric and Bessel functions

The next example employs the familar Taylor expansion of $\sin bx$. The result is expressed in terms of the *Bessel function of the first kind*

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{6.1}$$

Example 6.1. Let $c = b/8(1+a)$. Then

$$\int_0^\infty \sin\left(\frac{bx^2}{x^4+2ax^2+1}\right) dx = \frac{\pi b}{\sqrt{8(1+a)}} \left[J_0(2c)\cos 2c + J_1(2c)\sin 2c\right].$$

To verify this, apply Theorem 4.3 to $\sin bx$ to obtain

$$\int_0^\infty \sin\left(\frac{bx^2}{x^4+2ax^2+1}\right) dx = \frac{\pi b}{\sqrt{8(1+a)}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\binom{4k}{2k} c^{2k}.$$

Lemma 6.2. The following identity holds:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\binom{4k}{2k} c^{2k} = J_0(2c)\cos 2c + J_1(2c)\sin 2c.$$

Proof. Using the Cauchy product and the series expression (6.1), the right-hand side is written as

$$J_0(2c)\cos 2c + J_1(2c)\sin 2c =$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^k c^{2k} 4^j}{(2j)!(k-j)!^2} + 2c^2 \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^k c^{2k} 4^j}{(2j+1)!(k-j)!(k-j+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k c^{2k}(4k)!}{(2k)!^3} + 2c^2 \sum_{k=0}^{\infty} \frac{(-1)^k c^{2k}(4k+3)!}{(2k+3)!(2k+1)^2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\binom{4k}{2k} c^{2k}.
\end{aligned}$$

The passage from the first to the second equality is justified by the identities

$$\sum_{j=0}^{k} \frac{4^j}{(2j)!(k-j)!^2} = \frac{1}{(2k)!}\binom{4k}{2k} \quad\text{and}\quad \sum_{j=0}^{k} \frac{4^j}{(2j)!(k-j)!\,(k-j+1)!} = \frac{4k+3}{(2k+3)!}\binom{4k+2}{2k+1}.$$

Both of these formulas are in turn verifiable via the WZ-method [23] with their respective rational certificates

$$\frac{(6n^2+10n+4-4nk-34k)(2k-1)k}{(n+1-k)^2(4n+1)(4n+3)} \quad\text{and}\quad \frac{(20n+17+6n^2-4nk-7k)(2k-1)}{(n+1-k)(n+2-k)(4n+5)(4n+7)}.$$

□

Example 6.3. The choice $a=0$ and $b=1$ in Example 6.1 produces

$$\int_0^\infty \sin\left(\frac{x^2}{x^4+1}\right) dx = \frac{\pi}{2\sqrt{2}} \left[J_0\left(\tfrac{1}{4}\right)\cos\left(\tfrac{1}{4}\right) + J_1\left(\tfrac{1}{4}\right)\sin\left(\tfrac{1}{4}\right)\right]. \tag{6.2}$$

Example 6.4. By choosing $a = b = 1$ in Example 6.1, we get

$$\int_0^\infty \sin\left(\left[\frac{x}{x^2+1}\right]^2\right) dx = \frac{\pi}{4}\left[J_0\left(\tfrac{1}{8}\right)\cos\left(\tfrac{1}{8}\right) + J_1\left(\tfrac{1}{8}\right)\sin\left(\tfrac{1}{8}\right)\right]. \tag{6.3}$$

As the time of this writing, `Mathematica` is unable to evaluate the integrals in the two previous examples.

Note. The function

$$g(u) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}\binom{4k}{2k} u^{2k} \tag{6.4}$$

also has a hypergeometric form as

$$g(u) = {}_2F_3\left[\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2}\end{matrix}; -4u^2\right]. \tag{6.5}$$

This follows directly from the identity

$$\frac{\binom{4k}{2k}}{(2k+1)!} = \frac{2^{2k-3/2}\Gamma(k+1/4)\Gamma(k+3/4)}{\Gamma(k+1/2)\,\Gamma^2(k+1)\,\Gamma(k+3/2)},$$

that is established via the duplication formula of the gamma function

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}} 2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2})$$

and its iteration

$$\Gamma(4x) = \frac{1}{\pi\sqrt{\pi}} 2^{8x-5/2}\Gamma(x)\Gamma(x+\tfrac{1}{4})\Gamma(x+\tfrac{1}{2})\Gamma(x+\tfrac{3}{4})$$

Then $\Gamma(a+k) = (a)_k\Gamma(a)$ produces (6.5). Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. This gives an alternative form of the result described in Example 6.1, i.e.,

$$\int_0^\infty \sin\left(\frac{bx^2}{x^4+2ax^2+1}\right) dx = \frac{\pi\, b}{2\sqrt{2(1+a)}}\ {}_2F_3\left[\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2}\end{matrix}; -b^2/16(1+a)^2\right].$$

Note. Proceeding as in Example 2.3 we can obtain

$$\int_0^\infty (x^2+x^{-2}-1)\sin\left(\frac{x^6+x^{-6}-2}{x^{12}-4x^6-4x^{-6}+x^{-12}+7}\right) dx = \frac{\pi}{6\sqrt{2}}\left[J_0\left(\tfrac{1}{4}\right)\cos\tfrac{1}{4} + J_1\left(\tfrac{1}{4}\right)\sin\tfrac{1}{4}\right].$$

`Mathematica` is unable to compute the preceding integral.

Note. The result in Example 6.1 can also be established by the Frullani method described in Section 5. Start with

$$I := \int_0^\infty \sin\left(\frac{bx^2}{x^4+2ax^2+1}\right) dx = \frac{\sqrt{b}}{2\sqrt{a^*}}\int_0^1 \frac{\sin(a^*t)}{t}\frac{dt}{\sqrt{t(1-t)}}.$$

As before, write

$$I = \frac{\sqrt{a^*b}}{2}\int_0^1\int_0^1 \frac{\cos(a^*ty)}{\sqrt{t(1-t)}} dt\, dy.$$

The inner integral is a well-known cosine transform ([10],p.12, 1.3.14[2]

$$\int_0^1 \frac{\cos(\omega t)\,dt}{\sqrt{t(1-t)}} = \pi J_0\left(\frac{\omega}{2}\right)\cos\frac{\omega}{2}, \tag{6.6}$$

and it follows that

$$I = \frac{\pi\sqrt{b}}{\sqrt{a^*}} \int_0^{a^*/2} \cos t\, J_0(t)\,dt.$$

The result is now immediate from the identity

$$\frac{d}{dt}\left[t\left(\cos t\, J_0(t) + \sin t\, J_1(t)\right)\right] = \cos t\, J_0(t).$$

7. The sine integral and Bessel functions

The *sine integral* is defined by

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt. \tag{7.1}$$

The Cauchy-Schlömilch transformation can be employed to prove

$$\int_0^\infty \mathrm{Si}\left(\frac{bx^2}{x^4+2ax^2+1}\right)dx = \pi\sqrt{2(1+a)}\,\left[(4c\cos 2c - \sin 2c)J_0(2c) + 4c\sin 2c\, J_1(2c)\right],$$

where $c = b/8(1+a)$. To establish this identity, start with the evaluation in Example 6.1

$$\int_0^\infty \sin\left(\frac{bx^2}{x^4+2ax^2+1}\right)dx = \frac{\pi b}{\sqrt{8(1+a)}}\left[J_0(2c)\cos 2c + J_1(2c)\sin 2c\right]$$

divide by b and integrate both sides. Then the identity

$$\frac{d}{dx}\left[(2x\cos x - \sin x)J_0(x) + 2x\sin x J_1(x)\right] = J_0(x)\cos x + J_1(x)\sin x \tag{7.2}$$

gives the result.

Example 7.1. The special case $a = 0$ and $b = 1$ yields

$$\int_0^\infty \mathrm{Si}\left(\frac{x^2}{x^4+1}\right)dx = \frac{\pi}{2\sqrt{2}}\left[J_0(\tfrac{1}{4})\left[\cos\tfrac{1}{4} - 2\sin\tfrac{1}{4}\right] + J_1(\tfrac{1}{4})\sin\tfrac{1}{4}\right].$$

Example 7.2. The special case $a = b = 1$ yields

$$\int_0^\infty \mathrm{Si}\left(\left[\frac{x}{x^2+1}\right]^2\right)dx = \frac{\pi}{2}\left[J_0(\tfrac{1}{8})\left[\cos\tfrac{1}{8} - 4\sin\tfrac{1}{8}\right] + J_1(\tfrac{1}{8})\sin\tfrac{1}{8}\right].$$

`Mathematica` is unable to evaluate these integrals.

[2]Note that formula 1.3.14 in [10] is incorrect.

8. The Riemann zeta function

Interesting examples of definite integrals come from integral representations of special functions. For the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s}, \tag{8.1}$$

one such expression is given by

$$\zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s)} \int_0^\infty \frac{t^{s-1}\,dt}{1+e^t}. \tag{8.2}$$

Analytic properties of $\zeta(s)$ are often established via such integral formulas.

The change of variables $t = y^2$ produces

$$\int_0^\infty \frac{y^{2s-1}\,dy}{1+e^{y^2}} = \frac{1}{2}(1-2^{1-s})\Gamma(s)\zeta(s). \tag{8.3}$$

Example 8.1. We now employ Theorem 2.1 to establish

$$\int_0^\infty \frac{x^{2s+1}\,dx}{\cosh^2(x^2)} = 2^{-s}(1-2^{1-s})\Gamma(s+1)\zeta(s). \tag{8.4}$$

The notation

$$\Lambda(s) := \frac{1-2^{1-s}}{2}\Gamma(s)\zeta(s) \tag{8.5}$$

is employed in the proof. First introduce the change of variable $x = t^r$ in the Cauchy- Schlömilch formula and take $a = b$ for simplicity. Then

$$\int_0^\infty t^{r-1} f\left[a^2(t^r - t^{-r})^2\right]\,dt = \frac{1}{ar}\int_0^\infty f(y^2)\,dy. \tag{8.6}$$

Now let $f(x) = x^{s-1/2}/(1+e^x)$. Using the notation $S_r = \sinh(r\ln t)$, (8.6) yields

$$\frac{\Lambda(s)}{ar} = \int_0^\infty \frac{t^{r-1}(at-at^{-r})^{2s-1}\,dt}{1+\exp[(at^r-at^{-r})^2]} = \int_0^\infty \frac{t^{r-1}(2aS_r)^{2s-1}\,dt}{1+\exp[(2aS_r)^2]}. \tag{8.7}$$

Differentiate (8.7) with respect to a and use

$$\frac{c}{(1+c)^2} = \frac{1}{4\cosh^2(\theta/2)} \tag{8.8}$$

to obtain

$$\frac{2s-1}{a}\frac{\Lambda(s)}{ar} - 2(2a)^{2s-1}\int_0^\infty \frac{t^{r-1}S_r^{2s+1}\,dt}{\cosh^2[2(aS_r)^2]} = -\frac{\Lambda(s)}{a^2r}, \tag{8.9}$$

that produces

$$\int_0^\infty \frac{t^{r-1}S_r^{2s+1}\,dt}{\cosh^2[2(aS_r)^2]} = \frac{2s\Lambda(s)}{(2a)^{2s}a^2r}. \tag{8.10}$$

Now change t by t^{-1} in (8.10) and average the resulting integral with itself. The outcome is written as

$$\int_0^\infty \frac{t^{r-1}S_r^{2s+1}\,dt}{\cosh^2[2(aS_r)^2]} = \frac{2s\Lambda(s)}{(2a)^{2s}a^2r}. \tag{8.11}$$

The final change of variables $x = \sqrt{2}aS_r$ produces the stated result.

The special case $s = \frac{1}{2}$ yields

$$\int_0^\infty \frac{x^2\,dx}{\cosh^2(x^2)} = -\frac{1}{4}(2-\sqrt{2})\zeta(1/2)\sqrt{\pi}. \tag{8.12}$$

`Mathematica` is unable to produce (8.12).

9. The error function

Several entries in the table [13] involve the *error function*

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt. \tag{9.1}$$

For instance, entry 3.466.1 states that

$$\int_0^\infty \frac{e^{-\mu^2x^2}\,dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\left(1-\text{erf}(\mu\beta)\right)e^{\mu^2\beta^2}. \tag{9.2}$$

`Mathematica` is able to compute this example, which can be checked by writing the exponential in the integrand as $e^{\mu^2\beta^2} \times e^{-\mu^2(x^2+\beta^2)}$ and differentiating with respect to μ^2.

Example 9.1. The Cauchy-Schlömilch transformation is now applied to the function

$$f(x) = e^{-\mu^2x^2}/(x^2+2(a+1)) \tag{9.3}$$

to produce

$$\int_0^\infty \frac{e^{-\mu^2(x^2+x^{-2})}\,dx}{x^2+2a+x^{-2}} = \frac{\pi e^{2a\mu^2}}{2\sqrt{2(a+1)}}\left[1-\text{erf}\left(\mu\sqrt{2(a+1)}\,\right)\right]. \tag{9.4}$$

The choice $a = \mu = 1$ yields

$$\int_0^\infty \frac{e^{-(x^2+x^{-2})}\,dx}{(x+x^{-1})^2} = \frac{\pi e^2}{4}\left[1-\text{erf}(2)\right], \tag{9.5}$$

and $a = 0$, $\mu = 1$ gives

$$\int_0^\infty \frac{e^{-(x^2+x^{-2})}\,dx}{x^2+x^{-2}} = \frac{\pi}{2\sqrt{2}}\left[1-\text{erf}(\sqrt{2})\right]. \tag{9.6}$$

Neither of these special cases is computable by the current version of `Mathematica`.

10. Elliptic integrals

The classical *elliptic integral of the first kind* is defined by

$$\mathbf{K}(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{10.1}$$

The table [13] contains a variety of definite integrals that can be evaluated in terms of $\mathbf{K}(k)$. For instance, entry 3.843.4 states that

$$\int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2(2x)}}\frac{dx}{x} = \mathbf{K}(k). \tag{10.2}$$

The reader will find in [16] a large variety of examples.

In the context of the Cauchy-Schlömilch transformation, we present two illustrative examples.

Example 10.1. The first result is

$$\int_0^\infty \frac{x^2\,dx}{\sqrt{(x^4+2ax^2+1)(x^4+2bx^2+1)}} = \frac{1}{\sqrt{2(a+1)}}\mathbf{K}\left(\sqrt{\frac{a-b}{a+1}}\right). \tag{10.3}$$

To verify this result, apply (2.1) to the function

$$f(x) = \frac{1}{\sqrt{(x+2a+2)(x+2b+2)}}$$

and observe that

$$f\left((x-1/x)^2\right) = \frac{x^2}{\sqrt{(x^4+2ax^2+1)(x^4+2bx^2+1)}}.$$

The Cauchy-Schlömilch transformation gives

$$\int_0^\infty \frac{x^2\,dx}{\sqrt{(x^4+2ax^2+1)(x^4+2bx^2+1)}} = \int_0^\infty \frac{dx}{\sqrt{(x^2+A^2)(x^2+B^2)}}$$

with $A^2 = 2(a+1)$ and $B^2 = 2(b+1)$. This last integral is computed by the change of variable $x = A\tan\varphi$ and the trigonometric form of the elliptic integral yields (10.3).

Example 10.2. A similar argument produces the second elliptic integral evaluation. This time it involves

$$F(\varphi,k) := \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{10.4}$$

the (incomplete) elliptic integral of the first kind.

Assume $a \leq b \leq c$. Then

$$\int_0^\infty \frac{x^3\,dx}{\sqrt{(x^4+2ax^2+1)(x^4+2bx^2+1)(x^4+2cx^2+1)}} = \frac{1}{2\sqrt{(b+1)(c-a)}}F\left[\sin^{-1}\sqrt{\frac{c-a}{c+1}}, \sqrt{\frac{(b-a)(c+1)}{b+1)(c-a)}}\right]. \tag{10.5}$$

Following [12] a generalization of the Cauchy-Schlömilch identity is now used to evaluate some hyper-elliptic integrals.

Theorem 10.3. Assume $\phi(z)$ is a meromorphic function with only real simple poles a_j with $\text{Res}(\phi; a_j) < 0$. Moroever assume ϕ is asymptotically linear. Then, for any even real valued function f,

$$\int_0^\infty f\left[\phi(x)\right]\,dx = \int_0^\infty f(x)\,dx. \tag{10.6}$$

Example 10.4. As a simple illustration we take

$$\phi_N(z) = z\prod_{j=1}^N \frac{z^2-b_j^2}{z^2-a_j^2}, \tag{10.7}$$

where $a_1 < a_2 < \cdots < a_N$. Take $N = 1$ and write $b_1 = b$. Theorem 10.3 and

$$\int_0^\infty \frac{dx}{\sqrt{(x^2+\alpha^2)(x^2+\beta^2)}} = \frac{1}{\alpha}\mathbf{K}\left(\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right), \tag{10.8}$$

give

$$\int_0^\infty \frac{(t-b^2)^2\,dt}{\sqrt{tP(t)Q(t)}} = \frac{2}{\alpha}\mathbf{K}\left(\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\right), \tag{10.9}$$

with

$$\begin{aligned} P(t) &= t^3 + (\alpha^2 - 2a^2)t^2 + (a^4 - 2\alpha^2 b^2)t + \alpha^2 b^4, \\ Q(t) &= t^3 + (\beta^2 - 2a^2)t^2 + (a^4 - 2\beta^2 b^2)t + \beta^2 b^4. \end{aligned}$$

As an interesting special case, we have

$$\int_0^\infty \frac{(t-\frac{1}{2})^2\,dt}{\sqrt{t\,[t^3 - 2(kk')^2 t + k^2]\,[t^3 - 4(kk')^2 t^2 + k^4]}} = \frac{1}{k}\mathbf{K}'(k), \tag{10.10}$$

where k' is the complementary modulus and $\mathbf{K}'(k) := \mathbf{K}(k')$. `Mathematica` is unable to deal with this.

11. An extension of the Cauchy-Schlömilch method

An extension of Theorem 2.1 by Jones [14] is discussed here. The next section presents statistical applications of this result.

Theorem 11.1. Let s be a continuous decreasing function from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Assume f is self-inverse, that is, $s^{-1}(x) = s(x)$ for all $x \in \mathbb{R}^+$. Then

$$\int_0^\infty f\left([x - s(x)]^2\right)\,dx = \int_0^\infty f\left(y^2\right)\,dy, \tag{11.1}$$

provided the integrals are convergent.

Proof. The change of variables $t = s(x)$ yields

$$I = \int_0^\infty f([x-s(x)]^2)\,dx = -\int_0^\infty f([s(t)-t]^2)\,s'(t)\,dt. \tag{11.2}$$

The average of these two representations, followed by the change of variables $u = x - s(x)$ gives the result. ☐

Note. The above result is given without a scaling constant $a > 0$. This could be introduced by the change of variable $x_1 = ax$ in (11.1) to obtain, after relabeling x_1 as x,

$$\int_0^\infty f([ax - s(ax)]^2)\,dx = \frac{1}{a}\int_0^\infty f\left(y^2\right)\,dy. \tag{11.3}$$

Jones [14] lists several specific forms of self-inverse $s(x)$ along with two methods for generating such functions based on work of Kucerovsky, Marchand and Small [15].

Example 11.2. An attractive self-inverse function is

$$s(x) = x - \frac{1}{\alpha} \log\left(e^{\alpha x} - 1\right). \tag{11.4}$$

Then (11.1) becomes

$$\int_0^\infty f\left(\frac{1}{\alpha^2} \log^2(e^{\alpha x} - 1)\right) dx = \int_0^\infty f(y^2)\, dy. \tag{11.5}$$

The choice $f(x) = e^{-x}$ gives, using (3.1),

$$\int_0^\infty \exp\left(-\frac{1}{\alpha^2} \log^2(e^{\alpha x} - 1)\right) dx = \frac{\sqrt{\pi}}{2}. \tag{11.6}$$

Example 11.3. Several other examples of self-inverse functions are provided in Jones [14]. Each one produces a Cauchy-Schlömilch type integral. Some examples are

$$\begin{aligned}
\int_1^\infty f\left[(x - \exp(\alpha/\log x))^2\right] dx &= \int_0^\infty f(y^2)\, dy, \\
\int_0^\infty f\left[\frac{1}{\alpha^2} \log^2\left(\frac{e^{\alpha x}\sinh(\alpha x)}{1 + \cosh(\alpha x)}\right)\right] dx &= \int_0^\infty f(y^2)\, dy, \\
\int_0^\infty f\left[(x - \sinh(\alpha/\sinh^{-1} x))^2\right] dx &= \int_0^\infty f(y^2)\, dy.
\end{aligned}$$

12. Application to generating flexible probability distributions

There has recently been renewed interest in the statistical literature in generating flexible families of probability distributions for univariate continuous random variables. Baker [1] describes the use of Cauchy-Schlömilch transformation to generate new probability density functions from old ones. Jones [14] does the same with the extended transformation of Section 11. The identity (11.3) states that the total mass of $f(y^2)$ is the same as that of $af([ax - s(ax)]^2)$ for any self-inverse function s and any scaling constant $a > 0$.

There are many techniques for introducing one or more parameters into a simple 'parent distribution' with probability density function g, to produce more sophisticated distributions. One such method, not generally familiar, proceeds by 'transformation of scale', defining $f_b(x) \propto g(t_b(x))$ where $t_b(x)$ depends on the new parameter b. The difficulty associated with this procedure is the validation that f_b is integrable and then to explicitly provide its normalizing constant.

The Cauchy-Schlömilch result in Theorem 2.1 guarantees that the choice $t_b(x) = |x - bx^{-1}|$ produces from the density g of a positive random variable a new density f_b, also for a positive random variable, via

$$f_b(x) = g(|x - bx^{-1}|). \tag{12.1}$$

This was observed by Baker [1]. The parameter a in (2.1) is redundant for distribution theory work since its action as a scale parameter is well understood; a must, however, be reintroduced for practical fitting of such distributions to data.

A number of general properties of distributions with density of the form f_b follow, some of which are:

(i) f_b is R-symmetric [22] about R-center $\sqrt{b}$, i.e. $f_b(\sqrt{b}x) = f_b(\sqrt{b}x^{-1})$;

(ii) $f_b(0) = 0$ and

$$f_b(x) \approx g(b/x) \text{ as } x \to 0;\ \ f_b(x) \approx g(x) \text{ as } x \to \infty;$$

(iii) moment relationships follow from

$$E_{f_b}\{|X - bX^{-1})|^r\} = E_g(Y^r)$$

and, by R-symmetry,

$$E_{f_b}(X^r) = b^{r+1}E_{f_b}(X^{-(r+2)});$$

(iv) if g is decreasing, then f_b is unimodal with mode at $\sqrt{b}$;

(v) if f_b is unimodal, its mean and its median are both greater than its mode.

These properties can be found in Baker [1], but only special cases of (iii) are provided.

Amongst the most interesting distributions presented by Baker is the root-reciprocal inverse Gaussian distribution (RRIG). This example, also discussed in Mudholkar and Wang [22], in the case of dispersion parameter $\lambda = 1$, arises from (12.1) when g is the half-Gaussian density. The RRIG density is

$$f_b(x) = \sqrt{\frac{2}{\pi}}e^b \exp\left\{-\frac{1}{2}\left(x^2 + b^2/x^2\right)\right\}, \tag{12.2}$$

$b > 0$. This corresponds in integral terms to (3.3) above. Similarly, a second example presented by Baker [1] (Section 3.4) is the distribution based on the half-Subbotin distribution. This is directly linked to (3.5). A third example, based on the half-t distribution, has density

$$f_{\nu,b}(x) = \frac{2\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}(1 + (x - b/x)^2/\nu)^{-(\nu+1)/2}, \tag{12.3}$$

$\nu, b > 0$. The verification that $f_{\nu,b}(x)$ integrates to 1 can be done by using the integral I_3 in Theorem 4.1. Of course, other distributions in Baker [1] correspond to other integral formulae, while integral formulae in this article which have non-negative integrands correspond to other distributions.

Transformation of scale densities are particularly amenable to having their skewness assessed by the asymmetry function $\gamma(p)$, $0 < p < 1$, of Critchley and Jones [9] and provide relatively rare tractable examples thereof. Jones [14] shows that the asymmetry function associated with unimodal f_b is

$$\gamma_b(p) = \left(\sqrt{c_g^2(p) + 4b} - \sqrt{4b}\right)/c_g(p) \tag{12.4}$$

where $c_g(p) = g^{-1}(pg(0))$. This shows that the Cauchy-Schlömilch transformation of scale always results in positively skewed distributions, with asymmetry functions decreasing in p, which become more skew as b decreases. For example, for the RRIG density (12.2),

$$\gamma_b(p) = \left(\sqrt{2b - \log p} - \sqrt{2b}\right)/\sqrt{-\log p}. \tag{12.5}$$

The extended Cauchy-Schlömilch transformation described in Section 11 also affords new transformation of scale distributions, as explored by Jones [14]. Probability densities of the form

$$f_s(x) = g(|x - s(x)|) \tag{12.6}$$

for decreasing, onto, self-inverse s are described there. In this situation, properties (i)-(v) discussed above become:

(i′) f_s can be defined to be S-symmetric about S-center x_0 since $f_s(x) = f_s(s(x))$. Here x_0 is defined by $s(x_0) = x_0$;

(ii′) $f_s(0) = 0$ and

$$f_s(x) \approx g(s(x)) \text{ as } x \to 0; \;\; f_s(x) \approx g(x) \text{ as } x \to \infty;$$

(iii′) moment relationships follow from

$$E_{f_s}\{|X - s(X)|^r\} = E_g(Y^r)$$

and, by S-symmetry,

$$E_{f_s}(X^r) = -E_{f_s}(s'(X)s^r(X)).$$

A special case of the latter is that $E_{f_s}(s'(X)) = -1$;

(iv′) if g is decreasing, then f_s is unimodal with mode at x_0;

(v′) if f_s is unimodal and g is convex, its mean and its median are both greater than its mode.

By way of example, Jones [14] briefly explored the half-Gaussian-based analogue of (12.2) when $s(x)$ is given by (11.4). This has probability density

$$f_s(x) = \sqrt{\frac{2}{\pi}} \exp\left\{ -\frac{1}{2\alpha^2} \log^2\left(e^{\alpha x} - 1\right) \right\}. \tag{12.7}$$

The fact that (12.7) integrates to 1 is closely related to (11.6). Its asymmetry function has the form

$$\gamma_s(p) = \frac{1}{\alpha\sqrt{-(\log p)/2}} \log\left\{ \cosh\left(\alpha\sqrt{-(\log p)/2} \right) \right\}. \tag{12.8}$$

Like (12.5), this asymmetry function is always positive and decreases in p; (12.8) increases in α.

The Cauchy-Schlölmilch transformation has thus motivated and triggers a new and promising area of work in distribution theory.

13. Conclusions

The Cauchy-Schlömilch transformation establishes the equality of two definite integrals with integrands related in a simple manner. Applying this transformation to a variety of well-known definite integrals yields examples that are beyond the current capabilities of symbolic languages. Our purpose in this paper is not only to present the many integrals considered here, but also to give an exposition of the salient points of the Cauhy-Schlömilch transformation so as to serve as motivating examples to explore further symbolic integration algorithms.

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$$\int_0^1 \frac{t^{\mu+1/2}(1-t)^{\mu-1/2}}{(a+bt-ct^2)^{\mu+1}}.$$

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$$\int_0^1 \frac{t^{\mu+1/2}(1-t)^{\mu-1/2}}{(a+bt-ct^2)^{\mu+1}}.$$

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