

THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 4: THE GAMMA FUNCTION.

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ABSTRACT. We present a systematic derivation of some definite integrals in the classical table of Gradshteyn and Ryzik that can be reduced to the gamma function.

1. INTRODUCTION

The table of integrals [2] contains some evaluations that can be derived by elementary means from the *gamma function*, defined by

$$(1.1) \quad \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

The convergence of the integral in (1.1) requires $a > 0$. The goal of this paper is to present some of these evaluations in a systematic manner. The techniques developed here will be employed in future publications. The reader will find in [1] analytic information about this important function.

The gamma function represents the extension of factorials to real parameters. The value

$$(1.2) \quad \Gamma(n) = (n-1)!, \text{ for } n \in \mathbb{N}$$

is elementary. On the other hand, the special value

$$(1.3) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

is equivalent to the well-known *normal integral*

$$(1.4) \quad \int_0^\infty \exp(-t^2) dt = \frac{1}{2} \Gamma\left(\frac{1}{2}\right).$$

The reader will find in [1] proofs of Legendre's duplication formula

$$(1.5) \quad \Gamma\left(x + \frac{1}{2}\right) = \frac{\Gamma(2x)\sqrt{\pi}}{\Gamma(x)2^{2x-1}},$$

that produces for $x = m \in \mathbb{N}$ the values

$$(1.6) \quad \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}} \frac{(2m)!}{m!}.$$

This appears as **3.371** in [2].

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2. THE INTRODUCTION OF A PARAMETER

The presence of a parameter in a definite integral provides great amount of flexibility. In (1.1) there essentially a unique way to do this. The change of variables $x = \mu t$ yields

$$(2.1) \quad \Gamma(a) = \mu^a \int_0^\infty t^{a-1} e^{-\mu t} dt.$$

This appears as **3.381.4** in [2] and the choice $a = n + 1$, with $n \in \mathbb{N}$, that reads

$$(2.2) \quad \int_0^\infty t^n e^{-\mu t} dt = n! \mu^{-n-1}$$

appears as **3.351.3**.

The special case $a = m + \frac{1}{2}$, that appears as **3.371** in [2], yields

$$(2.3) \quad \int_0^\infty t^{m-\frac{1}{2}} e^{-\mu t} dt = \frac{\sqrt{\pi}}{2^{2m}} \frac{(2m)!}{m!} \mu^{-m-\frac{1}{2}},$$

is consistent with (1.6).

The combination

$$(2.4) \quad \int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}} dx = \frac{\mu^\rho - \nu^\rho}{\rho} \Gamma(1 - \rho),$$

that appears as **3.434.1** in [2] can now be evaluated directly. The integral **3.434.2**

$$(2.5) \quad \int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x} dx = \ln \frac{\nu}{\mu}$$

is obtained from (2.4) by passing to the limit as $\rho \rightarrow 0$. This is an example of *Frullani integrals* that will be discussed in a future publication.

The reader will be able to check **3.478.1**:

$$(2.6) \quad \int_0^\infty x^{\nu-1} \exp(-\mu x^p) dx = \frac{1}{p} \mu^{-\nu/p} \Gamma\left(\frac{\nu}{p}\right),$$

and **3.478.2**:

$$(2.7) \quad \int_0^\infty x^{\nu-1} [1 - \exp(-\mu x^p)] dx = -\frac{1}{|p|} \mu^{-\nu/p} \Gamma\left(\frac{\nu}{p}\right)$$

by introducing appropriate parameter reduction.

The parameters can be used to prove many of the classical identities for $\Gamma(a)$.

Proposition 2.1. The gamma function satisfies

$$(2.8) \quad \Gamma(a+1) = a \Gamma(a).$$

Proof. Differentiate (2.1) with respect to μ to produce

$$(2.9) \quad 0 = a \mu^{a-1} \int_0^\infty t^{a-1} e^{-\mu t} dt - \mu^a \int_0^\infty t^a e^{-\mu t} dt.$$

Now put $\mu = 1$ to obtain the result. □

Differentiating (1.1) with respect to the parameter a yields

$$(2.10) \quad \Gamma'(a) = \int_0^\infty x^{a-1} e^{-x} \ln x, dx.$$

Further differentiation introduces higher powers of $\ln x$:

$$(2.11) \quad \Gamma^{(n)}(a) = \int_0^\infty x^{a-1} e^{-x} (\ln x)^n dx.$$

In particular, for $a = 1$, we obtain:

$$(2.12) \quad \int_0^\infty (\ln x)^n e^{-x} dx = \Gamma^{(n)}(1).$$

The special case $n = 1$ yields

$$(2.13) \quad \int_0^\infty e^{-x} \ln x dx = \Gamma'(1).$$

The reader will find in [1], page 176 an elementary proof that $\Gamma'(1) = -\gamma$, where

$$(2.14) \quad \gamma := \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n$$

is Euler's constant. This is one of the fundamental numbers of Analysis.

On the other hand, differentiating (2.1) produces

$$(2.15) \quad \int_0^\infty x^{a-1} e^{-\mu x} (\ln x)^n dx = \left(\frac{\partial}{\partial a} \right)^n [\mu^{-a} \Gamma(a)],$$

that appears as **4.358.5** in [2]. Using Leibnitz's differentiation formula we obtain

$$(2.16) \quad \int_0^\infty x^{a-1} e^{-\mu x} (\ln x)^n dx = \mu^{-a} \sum_{k=0}^n (-1)^k \binom{n}{k} (\ln \mu)^k \Gamma^{(n-k)}(a).$$

In the special case $a = 1$ we obtain

$$(2.17) \quad \int_0^\infty e^{-\mu x} (\ln x)^n dx = \frac{1}{\mu} \sum_{k=0}^n (-1)^k \binom{n}{k} (\ln \mu)^k \Gamma^{(n-k)}(1).$$

The cases $n = 1, 2, 3$ appear as **4.331.1**, **4.335.1** and **4.335.3** respectively.

In order to obtain analytic expressions for the terms $\Gamma^{(n)}(1)$, it is convenient to introduce the *polygamma function*

$$(2.18) \quad \psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

The derivatives of ψ satisfy

$$(2.19) \quad \psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x),$$

where

$$(2.20) \quad \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}$$

is the *Hurwitz zeta function*. In particular this gives

$$(2.21) \quad \psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1).$$

The values of $\Gamma^{(n)}(1)$ can now be computed by recurrence via

$$(2.22) \quad \Gamma^{(n+1)}(1) = \sum_{k=0}^n \binom{n}{k} \Gamma^{(k)}(1) \psi^{(n-k)}(1),$$

obtained by differentiating $\Gamma'(x) = \psi(x)\Gamma(x)$.

Using (2.19) the reader will be able to check the first few cases of (2.15), we employ the notation $\delta = \psi(a) - \ln \mu$:

$$\begin{aligned} \int_0^\infty x^{a-1} e^{-\mu x} \ln^2 x \, dx &= \frac{\Gamma(a)}{\mu^a} \{ \delta^2 + \zeta(2, a) \}, \\ \int_0^\infty x^{a-1} e^{-\mu x} \ln^3 x \, dx &= \frac{\Gamma(a)}{\mu^a} \{ \delta^3 + 3\zeta(2, a)\delta - 2\zeta(3, a) \}, \\ \int_0^\infty x^{a-1} e^{-\mu x} \ln^4 x \, dx &= \frac{\Gamma(a)}{\mu^a} \{ \delta^4 + 6\zeta(2, a)\delta^2 - 8\zeta(3, a)\delta + 3\zeta^2(2, a) + 6\zeta(4, a) \}. \end{aligned}$$

These appear as **4.358.2**, **4.358.3** and **4.358.4**, respectively.

3. ELEMENTARY CHANGES OF VARIABLES

The use of appropriate changes of variables yields, from the basic definition (1.1), the evaluation of more complicated definite integrals. For example, let $x = t^b$ to obtain, with $c = ab - 1$,

$$(3.1) \quad \int_0^\infty t^c \exp(-t^b) \, dt = \frac{1}{b} \Gamma\left(\frac{c+1}{b}\right).$$

The special case $a = 1/b$, that is $c = 0$, is

$$(3.2) \quad \int_0^\infty \exp(-t^b) \, dt = \frac{1}{b} \Gamma\left(\frac{1}{b}\right),$$

that appears as **3.326.1** in [2]. The special case $b = 2$ is the normal integral (1.4).

We can now introduce an extra parameter via $t = s^{1/b}x$. This produces

$$(3.3) \quad \int_0^\infty x^m \exp(-sx^b) \, dx = \frac{\Gamma(a)}{s^a b},$$

with $m = ab - 1$. This formula appears (at least) three times in [2]: **3.326.2**, **3.462.9** and **3.478.1**. Moreover, the case $s = 1$, $c = (m + 1/2)n - 1$ and $b = n$ appears as **3.473**:

$$(3.4) \quad \int_0^\infty \exp(-x^n) x^{(m+\frac{1}{2})n-1} \, dx = \frac{(2m-1)!!}{2^m n} \sqrt{\pi}.$$

The form given here can be established using (1.6).

Differentiating (3.3) with respect to the parameter m (keeping in mind that $a = (m + 1)/b$), yields

$$(3.5) \quad \int_0^\infty x^m e^{-sx^b} \ln x \, dx = \frac{\Gamma(a)}{b^2 s^a} [\psi(a) - \ln s].$$

In particular, if $b = 1$ we obtain

$$(3.6) \quad \int_0^\infty x^m e^{-sx} \ln x \, dx = \frac{\Gamma(m+1)}{s^{m+1}} [\psi(m+1) - \ln s].$$

The case $m = 0$ and $b = 2$ gives

$$(3.7) \quad \int_0^\infty e^{-sx^2} \ln x \, dx = -\frac{\sqrt{\pi}}{4\sqrt{s}} (\gamma + \ln 4s),$$

where we have used $\psi(1/2) = -\gamma - 2 \ln 2$. This appears as **4.333** in [2].

An interesting example is $b = m = 2$. Using the values

$$(3.8) \quad \Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2 \text{ and } \psi\left(\frac{3}{2}\right) = 2 - 2 \ln 2 - \gamma$$

yields

$$(3.9) \quad \int_0^\infty x^2 e^{-sx^2} \ln x \, dx = \frac{1}{8s} (2 - \ln 4s - \gamma) \sqrt{\frac{\pi}{s}}.$$

The values of ψ at half-integers follow directly from (1.5). Formula (3.9) appears as **4.355.1** in [2]. Using (3.5) it is easy to verify

$$(3.10) \quad \int_0^\infty (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n},$$

and

$$(3.11) \quad \int_0^\infty (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}}.$$

These appear as, respectively, **4.355.3** and **4.355.4** in [2]. The term $(2n-1)!!$ is the semi-factorial defined by

$$(3.12) \quad (2n-1)!! = (2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1.$$

Finally, formula **4.369.1** in [2]

$$(3.13) \quad \int_0^\infty x^{a-1} e^{-\mu x} [\psi(a) - \ln x] \, dx = \frac{\Gamma(a) \ln \mu}{\mu^a}$$

can be established by the methods developed here. The more ambitious reader will check that

$$\int_0^\infty x^{n-1} e^{-\mu x} \left\{ [\ln x - \tfrac{1}{2}\psi(n)]^2 - \tfrac{1}{2}\psi'(n) \right\} \, dx = \frac{(n-1)!}{\mu^n} \left\{ [\ln \mu - \tfrac{1}{2}\psi(n)]^2 + \tfrac{1}{2}\psi'(n) \right\},$$

that is **4.369.2** in [2].

We can also write (3.5) in the exponential scale to obtain

$$(3.14) \quad \int_{-\infty}^\infty t e^{mt} \exp(-se^{bt}) \, dt = \frac{\Gamma(m/b)}{b^2 s^{m/b}} \left(\psi\left(\frac{m}{b}\right) - \ln s \right).$$

The special case $b = m = 1$ produces

$$(3.15) \quad \int_{-\infty}^\infty t e^t \exp(-se^t) \, dt = -\frac{(\gamma + \ln s)}{s}$$

that appears as **3.481.1**. The second special case, appearing as **3.481.2**, is $b = 2$, $m = 1$, that yields

$$(3.16) \quad \int_{-\infty}^\infty t e^t \exp(-se^{2t}) \, dt = -\frac{\sqrt{\pi}(\gamma + \ln 4s)}{4\sqrt{s}}.$$

This uses the value $\psi(1/2) = -(\gamma + 2 \ln 2)$.

There are many other possible changes of variables that lead to interesting evaluations. We conclude this section with one more: let $x = e^t$ to convert (1.1) into

$$(3.17) \quad \int_{-\infty}^{\infty} \exp(-e^x) e^{ax} dx = \Gamma(a).$$

This is **3.328** in [2].

As usual one should not prejudge the difficulty of a problem: the example **3.471.3** states that

$$(3.18) \quad \int_0^a x^{-\mu-1} (a-x)^{\mu-1} e^{-\beta/x} dx = \beta^{-\mu} a^{\mu-1} \Gamma(\mu) \exp\left(-\frac{\beta}{a}\right).$$

This can be reduced to the basic formula for the gamma function. Indeed, the change of variables $t = \beta/x$ produces

$$(3.19) \quad I = \beta^{-\mu} a^{\mu-1} \int_{\beta/a}^{\infty} (t - \beta/a)^{\mu-1} e^{-t} dt.$$

Now let $y = t - \beta/a$ to complete the evaluation. The table [2] writes μ instead of a : it seems to be a bad idea to have μ and u in the same formula, it leads to typographical errors that should be avoided.

Another simple change of variables gives the evaluation of **3.324.2**:

$$(3.20) \quad \int_{-\infty}^{\infty} e^{-(x-b/x)^{2n}} dx = \frac{1}{n} \Gamma\left(\frac{1}{2n}\right).$$

The symmetry yields

$$(3.21) \quad I = 2 \int_0^{\infty} e^{-(x-b/x)^{2n}} dx.$$

The change of variables $t = b/x$ yields, using $b > 0$,

$$(3.22) \quad I = 2b \int_0^{\infty} e^{-(t-b/t)^{2n}} \frac{dt}{t^2}.$$

The average of these forms produces

$$(3.23) \quad I = \int_0^{\infty} e^{-(x-b/x)^{2n}} \left(1 + \frac{b}{x^2}\right) dx.$$

Finally, the change of variables $u = x - b/x$ gives the result. Indeed, let $u = x - b/x$ and observe that u is increasing when $b > 0$. This restriction is missing in the table. Then we get

$$(3.24) \quad I = 2 \int_0^{\infty} e^{-u^{2n}} du.$$

This can now be evaluated via $v = u^{2n}$.

Note. In the case $b < 0$ the change of variables $u = x - b/x$ has an inverse with two branches, splitting at $x = \sqrt{-b}$. Then we write

$$(3.25) \quad \begin{aligned} I &:= 2 \int_0^\infty e^{-(x-b/x)^{2n}} dx \\ &= 2 \int_0^{\sqrt{-b}} e^{-(x-b/x)^{2n}} dx + 2 \int_{\sqrt{-b}}^\infty e^{-(x-b/x)^{2n}} dx. \end{aligned}$$

The change of variables $u = x - b/x$ is now used in each of the integrals to produce

$$(3.26) \quad I = 2 \int_{2\sqrt{-b}}^\infty \frac{u \exp(-u^{2n}) du}{\sqrt{u^2 + 4b}}.$$

The change of variables $z = \sqrt{u^2 + 4b}$ yields

$$(3.27) \quad I = 2 \int_0^\infty \exp(-(z^2 - 4b)^n) dz.$$

We are unable to simplify it any further.

4. THE LOGARITHMIC SCALE

Euler preferred the version

$$(4.1) \quad \Gamma(a) = \int_0^1 \left(\ln \frac{1}{u} \right)^{a-1} du.$$

We will write this as

$$(4.2) \quad \Gamma(a) = \int_0^1 (-\ln u)^{a-1} du,$$

for better spacing. Many of the evaluations in [2] follow this form. Section **4.215** in [2] consists of four examples: the first one, **4.215.1** is (4.1) itself. The second one, labeled **4.215.2** and written as

$$(4.3) \quad \int_0^1 \frac{dx}{(-\ln x)^\mu} = \frac{\pi}{\Gamma(\mu)} \operatorname{cosec} \mu\pi,$$

is evaluated as $\Gamma(1 - \mu)$ by (4.1). The identity

$$(4.4) \quad \Gamma(\mu)\Gamma(1 - \mu) = \frac{\pi}{\sin \pi\mu}$$

yields the second form. The reader will find in [1] a proof of this identity. The section concludes with the special values

$$(4.5) \quad \int_0^1 \sqrt{-\ln x} dx = \frac{\sqrt{\pi}}{2},$$

as **4.215.3** and **4.215.4**:

$$(4.6) \quad \int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}.$$

Both of them are special cases of (4.1).

The reader should check the evaluations **4.269.3**:

$$(4.7) \quad \int_0^1 \sqrt{-\ln x} x^{p-1} dx = \frac{1}{2} \sqrt{\frac{\pi}{p^3}},$$

and **4.269.4**:

$$(4.8) \quad \int_0^1 \frac{x^{p-1} dx}{\sqrt{-\ln x}} = \sqrt{\frac{\pi}{p}}$$

by reducing them to (2.1). Also **4.272.5**, **4.272.6** and **4.272.7**

$$(4.9) \quad \begin{aligned} \int_1^\infty (\ln x)^p \frac{dx}{x^2} &= \Gamma(1+p), \\ \int_0^1 (-\ln x)^{\mu-1} x^{\nu-1} dx &= \frac{1}{\nu^\mu} \Gamma(\mu), \\ \int_0^1 (-\ln x)^{n-\frac{1}{2}} x^{\nu-1} dx &= \frac{(2n-1)!!}{(2\nu)^n} \sqrt{\frac{\pi}{\nu}}, \end{aligned}$$

can be evaluated directly in terms of the gamma function.

Differentiating (4.1) with respect to a yields **4.229.4** in [2]:

$$(4.10) \quad \int_0^1 \ln(-\ln x) (-\ln x)^{a-1} dx = \Gamma'(a) = \psi(a)\Gamma(a),$$

with $\psi(a)$ defined in (2.18). The special case $a = 1$ is **4.229.1**:

$$(4.11) \quad \int_0^1 \ln(-\ln x) dx = -\gamma,$$

and

$$(4.12) \quad \int_0^1 \ln(-\ln x) \frac{dx}{\sqrt{-\ln x}} = -(\gamma + 2 \ln 2)\sqrt{\pi},$$

that appears as **4.229.3**, is obtained by using the values $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\psi(\frac{1}{2}) = -(\gamma + 2 \ln 2)$.

The same type of arguments confirms **4.325.11**

$$(4.13) \quad \int_0^1 \ln(-\ln x) \frac{x^{\mu-1} dx}{\sqrt{-\ln x}} = -(\gamma + \ln 4\mu)\sqrt{\frac{\pi}{\mu}},$$

and **4.325.12**:

$$(4.14) \quad \int_0^1 \ln(-\ln x) (-\ln x)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^\mu} \Gamma(\mu) [\psi(\mu) - \ln \nu].$$

In particular, when $\mu = 1$ we obtain **4.325.8**:

$$(4.15) \quad \int_0^1 \ln(-\ln x) x^{\nu-1} dx = -\frac{1}{\nu} (\gamma + \ln \nu).$$

5. THE PRESENCE OF FAKE PARAMETERS

There are many formulas in [2] that contain parameters. For example, **3.461.2** states that

$$(5.1) \quad \int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}}$$

and **3.461.3** states that

$$(5.2) \quad \int_0^\infty x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}}.$$

The change of variables $t = px^2$ eliminates the *fake* parameter p and reduces **3.461.2** to

$$(5.3) \quad \int_0^\infty t^{n-\frac{1}{2}} e^{-t} dt = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$$

and **3.461.3** to

$$(5.4) \quad \int_0^\infty t^n e^{-t} dt = n!.$$

These are now evaluated by identifying them with $\Gamma(n + \frac{1}{2})$ and $\Gamma(n + 1)$, respectively.

A second way to introduce fake parameters is to shift the integral (2.1) via $s = t + b$ to produce

$$(5.5) \quad \int_b^\infty (s-b)^{a-1} e^{-s\mu} ds = \mu^{-a} e^{-\mu b} \Gamma(a).$$

This appears as **3.382.2** in [2].

There are many more integrals in [2] that can be reduced to the gamma function. These will be reported in a future publication.

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