

# SEIZED OPPORTUNITIES

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*To Donald J. Newman, who knew that integrals were fun.*

## 1. INTRODUCTION

Every undergraduate student encounters the evaluation of integrals at an early stage of his/her education. In my case this happened in a class at the Universidad Santa Maria, Valparaiso, Chile. There, we aspiring engineers were required to use the CRC Table [25]. This note tells the story of a series of fortunate encounters that have introduced the author to the wonderful world of *the evaluation of integrals*. The reader will see that there are very interesting questions left even in apparently elementary parts of mathematics. Many of the results contained here are on the author's website <http://www.math.tulane.edu/~vhm>.

The main character of this paper is a sequence of rational numbers

$$(1.1) \quad d_{l,m} = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \quad m \in \mathbb{N} \text{ and } 0 \leq l \leq m,$$

that appeared in the evaluation of the quartic integral

$$(1.2) \quad N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.$$

This is a remarkable sequence, connected to many interesting questions. The recent advances in communications and the possibility of fast search on literature, has accelerated collaborations in mathematics. The study of the properties of  $\{d_{l,m}\}$ , has led the author to many rewarding and unexpected collaborations.<sup>1</sup>

## 2. THE EVALUATION OF INTEGRALS

Elementary mathematics leaves the impression that there is a marked difference between the two branches of calculus. *Differentiation* is a systematic subject: every evaluation is a consequence of a number of established rules and basic examples. However, *integration* is a mixture of art and science. The successful evaluation of an integral depends on the right approach, the right change of variables or a patient search in a table of integrals. In fact, the theory of *indefinite* integrals of elementary functions is complete [10]. Risch's algorithm determines whether a given function has an antiderivative within a given class of functions. For example, this theory shows that if  $f$  and  $g$  are rational functions with  $g(x)$  nonconstant, then  $f(x)e^{g(x)}$  has an elementary primitive precisely when  $f(x) = R'(x) + R(x)g'(x)$  for some rational function  $R$ . In particular  $e^{-x^2}$  has no elementary primitive – a well-known fact.

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<sup>1</sup>The author wishes to thank Dante V. Manna for remarks on an earlier version of the paper.

However, the theory of *definite* integrals is far from complete and there is no general theory available. The nature of the constant in the evaluation of a definite integral is hard to predict as seen in the example

$$(2.1) \quad \int_0^\infty e^{-x} dx = 1, \quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad \text{and} \quad \int_0^\infty e^{-x^3} dx = \Gamma\left(\frac{4}{3}\right).$$

The first integrand has an elementary primitive, the second integral is the classical Gaussian and the evaluation of the third requires Euler's *gamma function* defined by

$$(2.2) \quad \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

Interesting numbers emerge from elementary manipulations of integrals. To wit, differentiating (2.2) at  $a = 1$  yields the numerical constant

$$(2.3) \quad \int_0^\infty e^{-x} \log x dx = -\gamma$$

known as the *Euler's constant*, defined by  $\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right)$ . Havil's book [15] is devoted to the story of this intriguing constant.

Another illustration of the deceptiveness of definite integrals is the fact that

$$(2.4) \quad \int_{-\infty}^\infty \frac{dx}{(e^x - x - 1)^2 + \pi^2} = \frac{1}{2}$$

is obtained by elementary methods, but the similar-looking integral

$$(2.5) \quad \int_{-\infty}^\infty \frac{dx}{(e^x - x)^2 + \pi^2}$$

is given by  $(1 - W(1))^{-1}$ , where  $W(z)$  is the *Lambert W-function*, defined as the solution to the transcendental equation  $xe^x = z$ . It is unknown whether this integral has a simpler analytic representation, but experts believe it is unlikely that it does.

In this note, the reader will find some of the mathematics behind the evaluation of definite integrals. Most of the results are quite elementary, but be mindful if somebody asks you to compute an integral: if  $\zeta(s)$  denotes the classical Riemann zeta function, V. V. Volchkov [24] has shown that establishing the exact value

$$(2.6) \quad \int_0^\infty \frac{(1 - 12t^2)}{(1 + 4t^2)^3} \int_{1/2}^\infty \log |\zeta(\sigma + it)| d\sigma dt = \frac{\pi(3 - \gamma)}{32},$$

is equivalent to the Riemann hypothesis. *Evaluating (2.6) might be hard.*

It remains to explain *why we evaluate integrals*. This paper gives some anecdotal answers. The general response is that these questions lead to challenging problems that do not require an extensive background, which have provided inspiration for interesting student research projects [5, 6]. In addition, the computation of integrals has been shown to be connected to many parts of mathematics.

Once in a while, a nice evaluation produces a beautiful proof. For example,

$$(2.7) \quad \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi$$

proves that  $\pi \neq \frac{22}{7}$ . This evaluation, which has a long history, is used by H. Medina [20] to produce reasonable approximations to  $\tan^{-1} x$  and has been revisited by S. K. Lucas in [18]. The latter contains, among many interesting results, the identity

$$(2.8) \quad \int_0^1 \frac{x^5(1-x)^6(197+462x^2)}{530(1+x^2)} dx = \pi - \frac{333}{106}$$

that exhibits the relation of  $\pi$  to its second continued fraction approximation.

To explain our motivation, we adapt a quote from George Mallory, when asked about climbing the Everest. *We evaluate integrals because they are there.*

The mathematical point of view described here is the author's perspective of Experimental Mathematics. Supplementary accounts are given by D.H. Bailey and J. M. Borwein in [2] and also by D. Zeilberger in his interview [13].

### 3. A GRADUATE STUDENT

A version of this story has already been told in [21]. George Boros (1947-2003) came to my office one day, stating that he could evaluate the integral

$$(3.1) \quad N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.$$

His result says: for  $a > -1$  and  $m \in \mathbb{N}$ , we have

$$(3.2) \quad N_{0,4}(a; m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),$$

where  $P_m$  is a polynomial of degree  $m$ , written as

$$(3.3) \quad P_m(a) = \sum_{l=0}^m d_l(m) a^l,$$

and

$$d_{l,m} = \sum_{j=0}^l \sum_{s=0}^{m-l} \sum_{k=s+l}^m \frac{(-1)^{k-l-s}}{2^{3k}} \binom{2k}{k} \binom{2m+1}{2(s+j)} \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{l-j},$$

from which it follows that  $d_l(m)$  is a rational number.

The proof is elementary and is based on the changes of variables  $x = \tan \theta$ , and then George had the clever idea of *doubling the angle*; that is, introducing a new variable  $u = 2\theta$ . This yields a new form for the integral (3.1) and the expression for  $d_l(m)$ . The double angle substitution is the basic idea behind the new theory of *rational Landen transformations*. The reader will find in [19] a recent survey on this topic.

Having no experience in special functions, my reaction to this result was (i) a symbolic language like Mathematica or Maple must be able to do it, (ii) there must be a simpler formula for the coefficients  $d_{l,m}$  and (iii) it must be known.

It was surprising to find out that the Mathematica version of the time was unable to compute (3.1) when  $a$  and  $m$  are entered as parameters. The symbolic status of  $N_{0,4}(a; m)$  has not changed much since it was reported on [21]. Mathematica 6.2 is still unable to solve this problem.

On the other hand, the corresponding indefinite integral is evaluated in terms of the Appell-F1 function, defined by

$$(3.4) \quad F_1(a; b_1, b_2; c; x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m! n! (c)_{m+n}} x^m y^n,$$

as

$$\int \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = x F_1 \left[ \frac{1}{2}, 1+m, 1+m, \frac{3}{2}, -\frac{x^2}{a_+}, -\frac{x^2}{a_-} \right],$$

where  $a_{\pm} := a \pm \sqrt{-1 + a^2}$ . Here  $(a)_k = a(a+1) \cdots (a+k-1)$  is the ascending factorial. This clarifies my reaction (i) and also makes the point that the evaluation of integrals, with the help of a symbolic language is a natural guide into the field of Special Functions.

The search for a simpler formula started with the experimental observation that, in spite of the alternating signs in the formula for  $d_{l,m}$ , these coefficients are all positive. It took us some time to find

$$(3.5) \quad d_{l,m} = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

The first proof is based on the mysterious appearance of the integral  $N_{0,4}(a, m)$  in the expansion

$$(3.6) \quad \sqrt{a + \sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a; k-1) c^k.$$

George figured out how to use Ramanujan Master's Theorem [3] to produce (3.5). The author asked him many times to explain his train of thoughts leading to this connection. There was never a completely logical path: *He simply knew how to integrate.*

The expression

$$(3.7) \quad P_m(a) = 2^{-2m} \sum_{k=0}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k$$

shows that the polynomial  $P_m(a)$  is an example of the classical Jacobi family

$$(3.8) \quad P_m^{(\alpha, \beta)}(a) := \sum_{k=0}^m (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left( \frac{a+1}{2} \right)^k$$

with parameters  $\alpha = m + \frac{1}{2}$  and  $\beta = -(m + \frac{1}{2})$ . The parameters  $\alpha$  and  $\beta$ , usually constants, are now dependent upon  $m$ . We were surprised not to find an explicit evaluation for  $N_{0,4}(a; m)$  in [14]. It turns out that this integral appears in an equivalent form as entry 3.252.11. This is the answer to (iii).

#### 4. A CONFERENCE AT PENN STATE OR HOW I GOT ERDŐS NUMBER 2

It was then important to present our results in public. We decided to volunteer a talk at a conference. A special one celebrating Basil Gordon's 65<sup>th</sup> birthday was being organized at Penn State<sup>2</sup>. Trying to find a way to close my talk with a

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<sup>2</sup>The author wishes to use this occasion to thank the organizers for the chance to speak there.

question in number theory, it occurred to me to describe a new formula for  $d_{l,m}$ . The idea behind it is simple: write (3.2) as

$$(4.1) \quad P_m(a) = \frac{2}{\pi} [2(a+1)]^{m+\frac{1}{2}} N_{0,4}(a; m)$$

and compute  $d_{l,m}$  from the Taylor expansion at  $a = 0$  of the right hand side. This yields

$$(4.2) \quad d_{l,m} = \frac{1}{l!m!2^{m+l}} \left[ \alpha_l(m) \prod_{k=1}^m (4k-1) - \beta_l(m) \prod_{k=1}^m (4k+1) \right],$$

where  $\alpha_l$  and  $\beta_l$  are polynomials in  $m$  of degrees  $l$  and  $l-1$ , respectively. The last transparency from my talk contained the formula

$$(4.3) \quad d_{1,m} = \frac{1}{m!2^{m+1}} \left[ (2m+1) \prod_{k=1}^m (4k-1) - \prod_{k=1}^m (4k-1) \right],$$

and the observation that the numerator is an even number, so it might be of interest to find out the exact power of 2 that divides it, that is, its 2-adic valuation  $\nu_2(d_l(m))$ . (For a prime  $p$ , write  $m = p^a r$  where  $p$  does not divide  $r$ . Then the integer  $a$  is the  $p$ -adic valuation of  $m$ , denoted by  $\nu_p(m)$ ).

A short time later, I received a fax from Jeffrey Shallit stating that he had established the result

$$(4.4) \quad \nu_2(d_1(m)) = 1 - 2m + \nu_2 \left( \binom{m+1}{2} \right) + S_2(m),$$

where  $S_2(m)$  is the sum of the binary digits of  $m$ . *Revista Scientia* is a journal produced by the Department of Mathematics at Universidad Santa Maria, Valparaiso, Chile, my undergraduate institution. This was perfect timing: there was going to be a special issue dedicated to the memory of Miguel Blazquez, one of my undergraduate teachers. The results on the valuation of  $d_{1,m}$  appeared in [7].

The polynomials  $\alpha_l$  and  $\beta_l$  do not have simple analytic expressions. One uninspired day, we decided to compute their roots numerically. We were pleasantly surprised to discover the following:

**Theorem 4.1.** *For all  $l \geq 1$ , all roots of  $\alpha_l(m) = 0$  lie on the line  $\operatorname{Re} m = -\frac{1}{2}$ . Similarly, the roots of  $\beta_l(m) = 0$  for  $l \geq 2$  lie on the same vertical line.*

The first step in the proof of this theorem took place at lunch during the 2000 Summer Institute for Mathematics for Undergraduates at the University of Puerto Rico at Humacao. John Little was a guest speaker and he is enthusiastic about problems involving polynomials. The result of that conversation is a series of email exchanges where the details of the proof of Theorem 4.1 were explained to me. The location of the zeros of  $\alpha_l(m)$  now suggest studying the behavior of this family as  $l \rightarrow \infty$ . In the best of all worlds, one will obtain an analytic function of  $m$  with all the zeros on a vertical line. Perhaps some number theory will enter and ... *there is no telling what will happen.*

## 5. THE GRADSHTEYN AND RYZHIK PROJECT

The problem of analytic evaluations of definite integrals has been of interest to scientists since integral calculus was developed. The central question is, loosely stated:

Given a class of functions  $\mathfrak{F}$  and an interval  $[a, b] \subset \mathbb{R}$ , express the integral of  $f \in \mathfrak{F}$

$$I = \int_a^b f(x) dx,$$

in terms of the special values of functions from an enlarged class  $\mathfrak{G}$ .

The theory in the case of indefinite integrals is well developed. For instance, by elementary arguments it is possible to show that if  $\mathfrak{F}$  is the class of rational functions, then the enlarged class  $\mathfrak{G}$  is obtained by including logarithms and inverse trigonometric functions. In the 1980's G. Cherry discussed extensions of this classical paradigm. The following example illustrates the relevant issues in describing  $\mathfrak{G}$ : we can evaluate

$$(5.1) \quad \int \frac{x^3 dx}{\log(x^2 - 1)} = \frac{1}{2} \text{li}(x^4 - 2x^2 + 1) + \frac{1}{2} \text{li}(x^2 - 1),$$

but

$$(5.2) \quad \int \frac{x^2 dx}{\log(x^2 - 1)}$$

cannot be written in terms of elementary functions and the logarithmic integral

$$(5.3) \quad \text{li}(x) := \int \frac{dx}{\log x}$$

that appears in (5.1). The reader will find in [10] the complete theory behind integration in terms of elementary functions.

On the other hand, the theory of definite integrals is less developed. Examples are evaluated by a series of ad-hoc procedures and have been collected in tables. The earliest volume available to the author is *Tables d'integrales definies* [4], compiled by Bierens de Haan, who also presented in 1862 a survey of the methods employed in the verification of the entries. These tables form the main source for the popular volume by I. S. Gradshteyn and I. M. Ryzhik [14]. There are many other interesting tables of integrals, from the one by A. Apelblat, small and beautiful, to the five volume compendium by A. P. Prudnikov et al., encyclopedic and very expensive. The choice of [14] is a popular compromise.

Once the author realized that there were interesting mathematics encoded in formula 3.252.11 of [14] that gave (3.1), we began to wonder what else was in that table. Perhaps it would be a good idea to *verify every formula in it by hand*, since most entries cannot be evaluated symbolically. This has proven to be a larger task than originally thought. The author has begun a systematic verification of the entries in [14], and the proofs have appeared in *Revista Scientia*.

Given the large number of entries in [14], we have not yet developed an order in which to check them. Once in a while an entry catches our eye. This was the case with entry 3.248.5 in the sixth edition of the table by Gradshteyn and Ryzhik. The presence of the double square root in the appealing integral

$$(5.4) \quad \int_0^\infty \frac{dx}{(1+x^2)^{3/2} [\varphi(x) + \sqrt{\varphi(x)}]^{1/2}} = \frac{\pi}{2\sqrt{6}},$$

with

$$(5.5) \quad \varphi(x) = 1 + \frac{4x^2}{3(1+x^2)^2},$$

reminded us of (3.6). *Unfortunately* (5.4) *is incorrect*. The numerical value of the left hand side is approximately 0.666377 and the right hand side is about 0.641275. The table [14] is continually being revised. After we informed the editors of the error in 3.248.5, it was taken out. There is no entry 3.248.5 in [14]. At the present time, we are still reconcile this formula.

The revision of integral tables is nothing new. C. F. Lindman compiled in 1891 a long list of errors from the table by Bierens de Haan [4]. The editors of [14] maintain the webpage <http://www.mathtable.com/gr/> where the corrections to the table by I.S Gradshteyn and I. M. Ryzhik are stored.

Integral tables are organized like a phonebook: entries that *look* similar are placed together. However, the fact that two integrals are close in the table is not a reflection of the techniques involved in their evaluation. For example, 4.229.4 gives

$$(5.6) \quad \int_0^1 \log\left(\log\frac{1}{x}\right) \left(\log\frac{1}{x}\right)^{\mu-1} dx = \psi(\mu)\Gamma(\mu),$$

for  $\text{Re } \mu > 0$ , and 4.229.7 states that

$$(5.7) \quad \int_{\pi/4}^{\pi/2} \log \log \tan x dx = \frac{\pi}{4} \left( 4 \log \Gamma\left(\frac{3}{4}\right) - \log \pi \right).$$

Indeed, the formula (5.6) is established by the change of variables  $v = -\log x$  followed by differentiating the gamma function (2.2) with respect to the parameter  $\mu$ . The function  $\psi(\mu)$  in (5.6) is simply the logarithmic derivative of  $\Gamma(\mu)$  and the formula has been checked. The situation is quite different for (5.7). This formula is the subject of the elegant paper [23] in which the author uses analytic number theory to check its validity. The ingredients of the proof are quite formidable: I. Vardi shows that

$$(5.8) \quad \int_{\pi/4}^{\pi/2} \log \log \tan x dx = \frac{d}{ds} \Gamma(s)L(s) \Big|_{s=1},$$

where where  $L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots$  is the Dirichlet L-function. The computation of (5.8) is done in terms of the *Hurwitz zeta function*

$$(5.9) \quad \zeta(q, s) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s},$$

defined for  $0 < q < 1$  and  $\text{Re } s > 1$ .

Vardi's technique has been extended in Luis Medina's Ph.D. thesis at Tulane. Examples of integrals evaluated there include

$$(5.10) \quad \int_0^{\infty} \log x \log \tanh x dx = \frac{\gamma\pi^2}{8} - \frac{3}{4}\zeta'(2) + \frac{\pi^2 \log 2}{12},$$

and

$$(5.11) \quad \int_0^1 \log(1+x+x^2) \log \log 1/x \frac{dx}{x} = -\frac{\gamma\pi^2}{9} + \frac{1}{18}\pi^2 \log 3 + \frac{2}{3}\zeta'(2).$$

## 6. A PRODUCTIVE TRIP TO CHILE

During the summer of 1999 I was invited to lecture on Integrals at Universidad Santa Maria. During the presentation of Vardi's method using the Hurwitz zeta function to evaluate (5.6), Olivier Espinosa mentioned that this function plays a role in the problem of a gas of non-interacting electrons in the background of a uniform magnetic field. For instance, it is shown that the *density of states*  $g(E)$ , in terms of which all thermodynamic functions are to be computed, is written as

$$(6.1) \quad g(E) = V \frac{4\pi}{h^3} (2e\hbar B)^{1/2} E \mathcal{H}_{1/2} \left( \frac{E^2 - m^2}{2e\hbar B} \right),$$

where  $V$  stands for volume,  $B$  for magnetic field,  $m$  is the electron mass,  $\hbar$  is Planck's constant,  $e$  is the electron charge and

$$(6.2) \quad \mathcal{H}_z(q) := \zeta(z, \{q\}) - \zeta(z, q+1) - \frac{1}{2}q^{-z},$$

with  $\{q\}$  the fractional part of  $q$ . The Hurwitz zeta function also appears in the evaluation of functional determinants and many other parts of mathematical physics [12].

The function  $\log \Gamma(x)$  makes its appearance through Lerch's formula

$$(6.3) \quad \frac{d}{dz} \zeta(z, q) \Big|_{z=0} = \log \Gamma(q) - \log \sqrt{2\pi}.$$

The first few formulas are evaluated symbolically:

$$\begin{aligned} \int_0^1 q \log \Gamma(q) dq &= \frac{\zeta'(2)}{2\pi^2} + \frac{1}{3} \log \sqrt{2\pi} - \frac{\gamma}{12}, \\ \int_0^1 q^2 \log \Gamma(q) dq &= \frac{\zeta'(2)}{2\pi^2} + \frac{\zeta(3)}{4\pi^2} + \frac{1}{6} \log \sqrt{2\pi} - \frac{\gamma}{12}, \\ \int_0^1 q^3 \log \Gamma(q) dq &= \frac{\zeta'(2)}{2\pi^2} + \frac{3\zeta(3)}{8\pi^2} - \frac{3\zeta'(4)}{4\pi^4} + \frac{1}{10} \log \sqrt{2\pi} - \frac{3\gamma}{40}. \end{aligned}$$

My favorite unevaluated integral is, without a doubt, one that is related to Euler's result

$$(6.4) \quad \int_0^1 \log \Gamma(q) dq = \log \sqrt{2\pi}.$$

Using Lerch's formula and an expression for the product of two Hurwitz zeta functions, we obtained

$$(6.5) \quad \int_0^1 \log^2 \Gamma(q) dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{1}{3} \gamma \log \sqrt{2\pi} + \frac{4}{3} \log^2 \sqrt{2\pi} - (\gamma + 2 \log \sqrt{2\pi}) \frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}.$$

The obvious next step would be to evaluate

$$(6.6) \quad L_3 := \int_0^1 \log^3 \Gamma(q) dq.$$



We have been unable to do this, but this question has interesting connections with *multiple zeta values* of the form

$$(6.7) \quad T(a, b, c) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^a m^b (n+m)^c}.$$

The book [8] has a nice introduction to these sums.

This encounter in Chile began a fruitful collaboration. Olivier, who studies particle physics for a living, now spends his free time thinking about integrals. *Evaluating integrals will take you to unexpected places.*

## 7. COMBINATORIAL ASPECTS OF THE COEFFICIENTS $d_{l,m}$

Now we return to the coefficients  $d_{l,m}$  in (3.5). Fixing  $m$  and plotting the list  $\{d_{l,m} : 0 \leq l \leq m\}$  reveals their unimodality. Recall that a finite sequence of real numbers  $\{x_0, x_1, \dots, x_m\}$  is said to be *unimodal* if there exists an index  $m^*$  such that  $x_j$  increases up to  $j = m^*$  and decreases from then on, that is,  $x_0 \leq x_1 \leq \dots \leq x_{m^*}$  and  $x_{m^*} \geq x_{m^*+1} \geq \dots \geq x_m$ . A polynomial is said to be unimodal if its sequence of coefficients is unimodal. Unimodal polynomials arise often in combinatorics, geometry and algebra.

The unimodality of the coefficients  $d_{l,m}$  follows directly from the representation (3.7) and the next theorem.

**Theorem 7.1.** If  $P(x)$  is a polynomial with positive nondecreasing coefficients, then  $P(x+1)$  is unimodal.

A condition stronger than unimodality is *logconcavity*. A sequence of positive real numbers  $\{x_0, x_1, \dots, x_m\}$  is said to be *logarithmically concave* (or *logconcave* for short) if  $x_{j+1}x_{j-1} \geq x_j^2$  for  $1 \leq j \leq m-1$ . It is easy to see that if a sequence is logconcave then it is unimodal. Extensive computations showed that the sequence  $\{d_{l,m} : 0 \leq l \leq m\}$  was logconcave. This question lead us to the study of the zeros of the polynomial  $P_m(a)$ . It turns out that if all the zeros of a polynomial are real and negative, then it is logconcave and therefore unimodal. Unfortunately  $P_m$  has the minimal possible number of real zeros: 0 if  $m$  is even and 1 if odd. Figure 1 plots these zeros for  $1 \leq m \leq 100$  and Figure 2 plots the zeros of  $P_m$  divided by the corresponding degree.

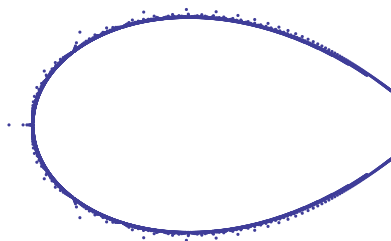


FIGURE 1. The zeros of  $P_m(a)$

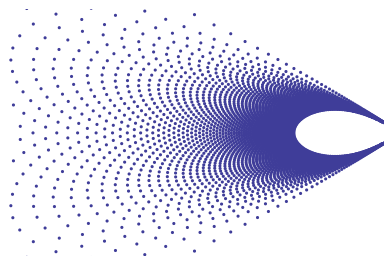


FIGURE 2. The scaled zeros of  $P_m(a)$

A remarkable result of Dimitrov [11] shows that the zeros of  $P_m(a)$  divided by the degree  $m$  converge to the left half of the lemniscate of Bernoulli given by the polar

equation  $r^2 = 2 \cos 2\theta$ , for  $\theta \in (3\pi/4, 5\pi/4)$ . This is reminiscent of the phenomena observed by Polya for the zeros of the partial sums of the exponential function.

The unimodality of  $d_{l,m}$  was relatively easy to show. The fact that  $d_{l,m}$  was logconcave turned out to be considerably more difficult and its proof came from an unexpected source [16]. Starting with the triple sum

$$(7.1) \quad d_{l,m} = \sum_{j,s,k} \frac{(-1)^{k+j-l}}{2^{3(k+s)}} \binom{2m+1}{2s} \binom{m-s}{k} \binom{2(k+s)}{k+s} \binom{s}{j} \binom{k}{l-j},$$

the authors used the RISC package MultiSum to produce the recurrence

$$(7.2) \quad 2(m+1)d_{l,m+1} = 2(l+m)d_{l-1,m} + (2l+4m+3)d_{l,m}$$

that implies the positivity of  $d_{l,m}$ . The next recurrence derived in automatic fashion is

$$(7.3) \quad (m+2-l)(m+l-1)d_{l-2,m} - (l-1)(2m+1)d_{l-1,m} + (l-1)ld_{l,m} = 0.$$

This enabled them to identify  $P_m(a)$  as a Jacobi polynomial. Finally, using the method of cylindrical algebraic decomposition, the authors produced the inequality

$$(7.4) \quad d_{l,m+1} \geq \frac{4m^2 + 7m + l + 3}{2(m+1-l)(m+1)} d_{l,m}$$

that implies the logconcavity of  $d_{l,m}$ .

Define the operator  $\mathfrak{L}(a_j) := a_j^2 - a_{j-1}a_{j+1}$ , so that a logconcave sequence  $\mathbf{a}$  is one such that  $\mathbf{a} := \{a_n : n \in \mathbb{N}\}$  and  $\mathfrak{L}(\mathbf{a})$  are positive. A sequence is called *infinitely logconcave* if it remains positive after applying  $\mathfrak{L}$  any number of times. We have conjectured that  $\{d_{l,m} : 0 \leq l \leq m\}$  is infinitely logconcave. The same seems to be true for the binomial coefficients  $\{\binom{m}{l} : 0 \leq l \leq m\}$ . Most likely this is easier to prove, but we have not been able to do so.

## 8. THE $p$ -ADIC POINT OF VIEW

In this last section we go back to divisibility questions for the sequence  $d_{l,m}$ . The generalization of (4.4) was obtained in joint work with T. Amdeberhan and D. Manna during the post-Katrina semester<sup>3</sup> when we were all on sabbatical out of necessity.

It is convenient to introduce some rescaling of  $d_{l,m}$  given by

$$(8.1) \quad A_{l,m} := l!m!2^{m+l}d_{l,m} = \frac{l!m!}{2^{m-l}} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

The pictures of the 2-adic valuations of  $A_{l,m}$  become increasingly complicated as  $l$  increases. Figure 3 shows  $l = 3$  and Figure 4 shows  $l = 59$ .

It was surprising to find out that the valuation of  $A_{l,m}$  is intimately linked to the Pochhammer symbol  $(a)_k = a(a+1)\cdots(a+k-1)$  in a very simple manner.

**Theorem 8.1.** The 2-adic valuation of  $A_{l,m}$  satisfies

$$(8.2) \quad \nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l.$$

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<sup>3</sup>The author wishes to thank the Courant Institute for its hospitality during that period.

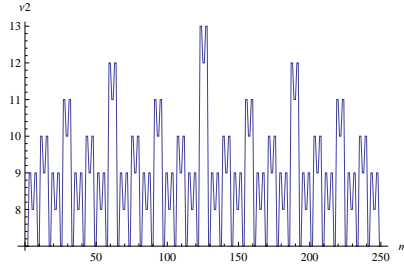


FIGURE 3. The valuation of  $A_{3,m}$

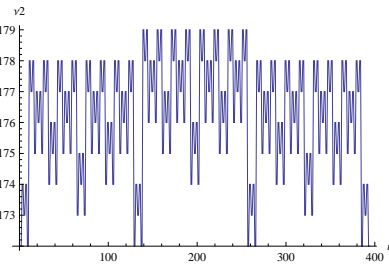


FIGURE 4. The valuation of  $A_{59,m}$

This result is now obtained in completely automatic fashion. Define the numbers

$$(8.3) \quad B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}.$$

It is required to prove that  $B_{l,m}$  is odd. The WZ-method [22] provides the recurrence

$$(8.4) \quad B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

Since the initial values  $B_{m,m} = 1$  and  $B_{m-1,m} = 2m+1$  are odd, it follows that  $B_{l,m}$  is an odd integer. There is also a genuine computer-free proof of this result. The point of view of the author is that we use all the tools available to us. *Experimenting with the computer is here to stay.*

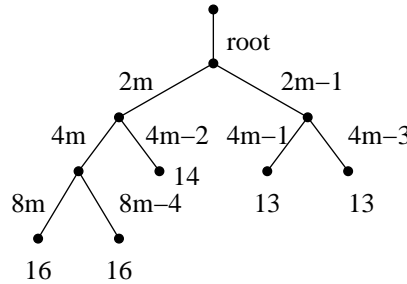


FIGURE 5. The decision tree for  $l = 5$

In view of the complexities seen in Figures 3 and 4 it was a remarkable surprise when Xinyu Sun told me that he had an *exact* formula for the 2-adic valuation of  $A_{l,m}$ . To describe it, we associate to each index  $l$  a labelled binary tree  $T(l)$  that encodes the 2-adic information of  $A_{l,m}$ . This is the *decision tree* for  $l$ . It is sufficient to consider  $l$  odd. Vertices of degree 1 will be called *terminal*. The description of  $T(l)$  is remarkably simple. The first generation of  $T(l)$  that contains terminal vertices is given by  $k^*(l) = \lfloor \log_2 l \rfloor$  and there are precisely  $2^{k^*+1} - l$  terminal vertices there. The tree  $T(l)$  has one more generation consisting of  $2(l - 2^{k^*})$  terminal vertices. There is also a well defined mechanism to label the terminal vertices (involving valuations of factorials).

The explicit formula for  $\nu_2(A_{5,j})$  is given by

$$(8.5) \quad \nu_2(A_{5,2j}) = \begin{cases} 14 + \nu_2\left(\frac{j+2}{4}\right) & \text{if } j \equiv 2 \pmod{4}, \\ 13 + \nu_2\left(\frac{j+1}{4}\right) & \text{if } j \equiv 3 \pmod{4}, \\ 13 + \nu_2\left(\frac{j+3}{4}\right) & \text{if } j \equiv 1 \pmod{4}, \\ 16 + \nu_2\left(\frac{j}{8}\right) & \text{if } j \equiv 0 \pmod{8}, \\ 16 + \nu_2\left(\frac{j+4}{8}\right) & \text{if } j \equiv 4 \pmod{8}, \end{cases}$$

for even indices. The odd index case is obtained from the relation  $\nu_2(A_{5,2j+1}) = \nu_2(A_{5,2j})$ .

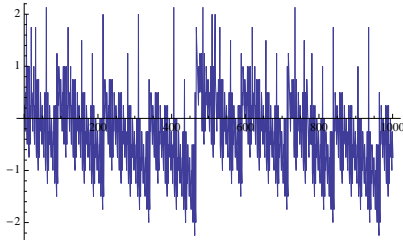


FIGURE 6. The error term  $\nu_5(A_{3,m}) - m/4$

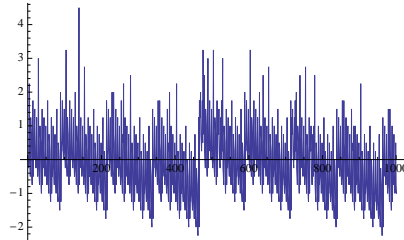


FIGURE 7. The error term  $\nu_5(A_{4,m}) - m/4$

The analysis for the prime 2 seems rather complete, but what about odd primes? A symbolic calculation shows that  $\nu_p(A_{l,m})$  grows linearly with  $m$ . Moreover, the slope is conjectured to be  $1/(p-1)$ . The error term for  $p=5$  is shown in Figures 6 and 7 for  $A_{l,m}$  with  $l=3$  and  $l=4$ , respectively. An analytic description might produce some more insight into this sequence.

The question of evaluation of definite integrals has taken us into a journey full of mathematical surprises. Many of them would not have been possible without the help of a symbolic software. We conclude with figures illustrating two instances of Experimental Mathematics:

(1) The presence of the function  $S_2(n)$  in formula (4.4) led us to work of T. Lengyel [17] on the 2-adic valuation of Stirling numbers of the second kind  $S(n, k)$ . These numbers have been around for a long time, so we expected everything to be known about them. The next four figures show a small sample of the graph of  $\nu_2(S(n, k))$  with  $k$  fixed. The case  $k=5$  has been analyzed [1], but the problem for  $k \geq 6$  is completely open.

(2) The special case  $p=3$  of the sequence

$$(8.6) \quad T_p(n) := \prod_{j=0}^{n-1} \frac{(pj+1)!}{(n+j)!},$$

appears as the number of  $n$  by  $n$  alternating sign matrices. The wonderful book [9] tells the story of this sequence. A seminar at Tulane devoted to this question led to the exploration of  $p$ -adic properties of these numbers. Figure 12 shows the graph of the 2-adic valuation of  $T_3(n)$  and Figure 13 the corresponding 3-adic valuation. The structure observed in these graphs is now beginning to be explored. There

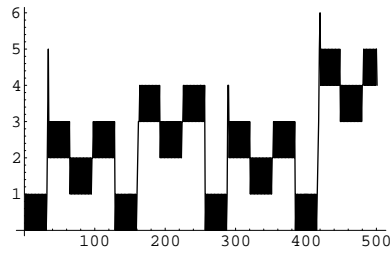


FIGURE 8. 2-adic valuation of  $S(n, 195)$

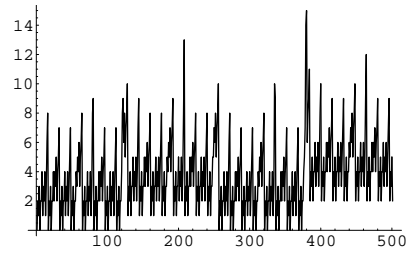


FIGURE 9. 2-adic valuation of  $S(n, 279)$

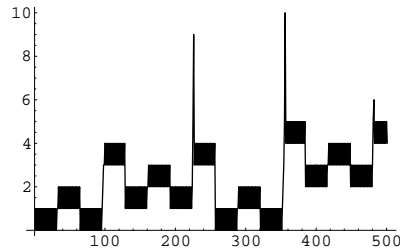


FIGURE 10. 2-adic valuation of  $S(n, 324)$

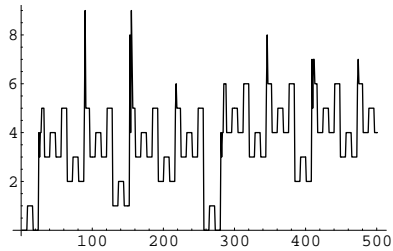


FIGURE 11. 2-adic valuation of  $S(n, 465)$

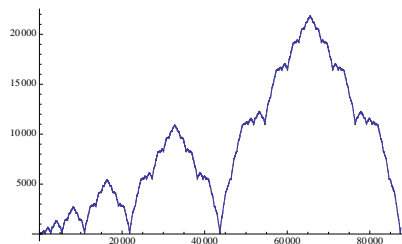


FIGURE 12. 2-adic valuation of  $T_3(n)$

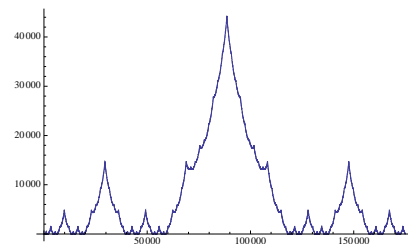


FIGURE 13. 3-adic valuation of  $T_3(n)$

are many interesting questions regarding  $T_p(n)$ , we leave the reader with the most natural one: *what do these numbers count?*

During the social parts at mathematical gatherings, the most common beginning of conversations is: *what do you do?* The author frequently encounters surprised faces when he states: *I compute integrals.* Perhaps this note has provided the reader a clearer response.

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