A symbolic approach to multiple zeta values at the negative integers

Victor H. Moll* [a], Lin Jiu [a] Christophe Vignat [a,b]
[a] *Department of Mathematics, Tulane University, New Orleans, USA*
* *Corresponding author*
[b] *LSS/Supelec, Université Orsay Paris Sud, France*

Reçu le *****; accepté après révision le +++++
Présenté par

Résumé

Une approche symbolique des valeurs de la fonction zêta multiple aux entiers négatifs Nous utilisons des techniques de calcul symbolique afin d'obtenir une expression simple d'une continuation analytique de la fonction d'Euler-Zagier, telle qu'elle est proposée dans la récente publication [1]. Cette approche nous permet de calculer des identités de recurrence sur les fonctions génératrices ainsi que sur la profondeur de la fonction zêta. *Pour citer cet article : A. Nom1, A. Nom2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Abstract

Symbolic computation techniques are used to derive some closed form expressions for an analytic continuation of the Euler-Zagier zeta function evaluated at the negative integers as recently proposed in [1]. This approach allows to compute explicitly some contiguity identities, recurrences on the depth of the zeta values and generating functions. *To cite this article: A. Nom1, A. Nom2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1. Introduction

The multiple zeta functions, first introduced by Euler and generalized by D. Zagier [2], appear in diverse areas such as quantum field theory [5] and knot theory [7]. These are defined by

$$\zeta_r(n_1,\dots,n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\dots k_r^{n_r}}, \tag{1}$$

Email addresses: vhm@tulane.edu (Victor H. Moll*), ljiu@tulane.edu (Lin Jiu), cvignat@tulane.edu (Christophe Vignat).

where $\{n_i\}$ are complex values, and (1) converges when the constraints

$$Re\left(n_r\right) \geq 1, \text{ and } \sum_{j=1}^{k} Re\left(n_{r+1-j}\right) \geq k, \quad 2 \leq k \leq r, \tag{2}$$

are satisfied (see [8]). Their values at integer points $\mathbf{n} = (n_1, \dots, n_r)$ satisfying (2) are called *multiple zeta values*. An equivalent definition of these values is

$$\zeta_r\left(n_1, \dots, n_r\right) = \sum_{k_1>0,\dots,k_r>0} \frac{1}{k_1^{n_1}\left(k_1+k_2\right)^{n_2} \dots \left(k_1+\dots+k_r\right)^{n_r}}.$$

The sum of the exponents $n_1 + \dots + n_r$ is called the *weight* of the zeta value, and the number r of these exponents is called its *depth*.

Following the result by Zhao [8] that the multiple zeta function has an analytic continuation to the whole space $\mathbb{C}^r$, several authors have recently proposed different analytic continuations based on a variety of approaches: Akiyama et al. [3] used the Euler-Maclaurin summation formula and Matsumoto [4] the Mellin-Barnes integral formula.

B. Sadaoui [1] provided recently such analytic continuation based on Raabe's identity, which links the multiple integral

$$Y_{\mathbf{a}}\left(\mathbf{n}\right) = \int_{[1,+\infty)^r} \frac{d\mathbf{x}}{\left(x_1+a_1\right)^{n_1}\left(x_1+a_1+x_2+a_2\right)^{n_2} \dots \left(x_1+a_1+\dots+x_r+a_r\right)^{n_r}}$$

to the multiple zeta function

$$Z\left(\mathbf{n},\mathbf{z}\right) = \sum_{k_1\geq1,\dots,k_r\geq1} \frac{1}{\left(k_1+z_1\right)^{n_1}\left(k_1+z_1+k_2+z_2\right)^{n_2} \dots \left(k_1+z_1+\dots+k_r+z_r\right)^{n_r}}$$

by

$$Y_{\mathbf{0}}\left(\mathbf{n}\right) = \int_{[0,1]^r} Z\left(\mathbf{n},\mathbf{z}\right) d\mathbf{z}.$$

B. Sadaoui uses a classical inversion argument to obtain an analytic continuation of the multiple zeta function defined at negative integer arguments $-\mathbf{n} = \left(-n_1, \dots, -n_r\right)$. The argument uses the following three steps:

- the integral $Y_{\mathbf{a}}\left(\mathbf{n}\right)$ is computed for values of $n_1, \dots, n_r$ that satisfy the convergence conditions (2),
- the values $\mathbf{n}$ are replaced by $-\mathbf{n}$ in this result: it is then shown that $Y_{\mathbf{a}}\left(-\mathbf{n}\right)$ is a polynomial in the variable $\mathbf{a}$,
- the variables $\mathbf{a} = (a_1, \dots, a_r)$ are replaced by $\left(\mathcal{B}_1, \dots, \mathcal{B}_r\right)$, and each Bernoulli symbol $\mathcal{B}_k$ satisfies the two evaluation rules:

evaluation rule 1: each power $\mathcal{B}_k^p$ of the Bernoulli symbol $\mathcal{B}_k$ should be evaluated as

$$\mathcal{B}_k^p \to B_p, \tag{3}$$

the $p-$th Bernoulli number

evaluation rule 2: for two different symbols $\mathcal{B}_k$ and $\mathcal{B}_l,\ k \neq l$, the product $\mathcal{B}_k^p \mathcal{B}_l^q$ is evaluated as

$$\mathcal{B}_k^p \mathcal{B}_l^q \to B_p B_q, \tag{4}$$

the product of the Bernoulli numbers B_p and B_q. If $k = l$, the first rule applies to give the evaluation

$$\mathcal{B}_k^p \mathcal{B}_k^q \to B_{p+q}.$$

Example 1. An example of depth 2, appearing in [1], is now computed using the rules above. The integral $Y_{\mathbf{a}}(n_1,n_2)$ is explicitly computed and, replacing (n_1,n_2) by $(-n_1,-n_2)$ gives

$$Y_{a_1,a_2}(-n_1,-n_2)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}a_1^{l_1}a_2^{l_2}.$$

Then substituting the variables a_1 and a_2 with the Bernoulli symbols $\mathcal{B}_1$ and $\mathcal{B}_2$ gives

$$\zeta_2(-n_1,-n_2)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}\mathcal{B}^{l_1}\mathcal{B}^{l_2}.$$

Using the evaluation rules (3) and (4) for the Bernoulli symbols, the multiple zeta value of depth 2 at $(-n_1,-n_2)$ is

$$\zeta_2(-n_1,-n_2)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}B_{l_1}B_{l_2}.$$

The general case is given in [1, eq. (4.10)] as the $(2r-1)$−fold sum [1]

$$\zeta_r(-n_1,\dots,-n_r)=(-1)^r\sum_{k_2,\dots,k_r}\frac{1}{(\bar{n}+r-\bar{k})}\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)} \times\sum_{l_1,\dots,l_r}\binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\dots\binom{k_r}{l_r}B_{l_1}\dots B_{l_r} \tag{5}$$

where $k_2,\dots,k_r\geq 0$, $l_j\leq k_j$ for $2\leq j\leq r$ and $l_1\leq\bar{n}+r+\bar{k}$ and

$$\bar{n}=\sum_{j=1}^{r}n_j,\ \bar{k}=\sum_{j=2}^{r}k_j. \tag{6}$$

A symbolic expression for (5) is proposed here. This is used as a convenient tool to derive some specific zeta values at negative integers, contiguity identities for the multiple zeta functions, recursions on their depth and generating functions.

2. Main result

Introduce first the symbols $\mathcal{C}_{1,2,\dots,k}$ defined recursively in terms of the Bernoulli symbols $\mathcal{B}_1,\dots,\mathcal{B}_r$ as

$$\mathcal{C}_1^n=\frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n=\frac{(\mathcal{C}_1+\mathcal{B}_2)^n}{n},\dots\ \text{and}\ \mathcal{C}_{1,2,\dots,k+1}^n=\frac{(\mathcal{C}_{1,2,\dots,k}+\mathcal{B}_{k+1})^n}{n}$$

with the symbolic computation rule:

$\mathcal{C}$−**symbols rule**: All symbols $\mathcal{C}_{1,2,\dots,k}$ are expanded using the above identities to express them only in terms of $\mathcal{B}_k$. The evaluation rules (3) and (4) for the Bernoulli symbols are then applied.

Example 2. For example,

1. This corrects a typo in [1, eq. (4.10)]

$$\mathcal{C}_1^{n_1}\mathcal{C}_2^{n_2} = \mathcal{C}_1^{n_1}\frac{(\mathcal{C}_1+\mathcal{B}_2)^{n_2}}{n_2} = \frac{1}{n_2}\sum_{k=0}^{n_2}\binom{n_2}{k}\mathcal{C}_1^{n_1+k}\mathcal{B}_2^{n_2-k} = \frac{1}{n_2}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{\mathcal{B}_1^{n_1+k}}{n_1+k}\mathcal{B}_2^{n_2-k}$$

is evaluated as

$$\frac{1}{n_2}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{B_{n_1+k}}{n_1+k}B_{n_2-k}.$$

The next result is given in terms of this notation.

Theorem 2.1 *The multiple zeta values (5) at the negative integers $(-n_1,\dots,-n_r)$ are given by*

$$\zeta_r(-n_1,\dots,-n_r) = \prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\dots,k}^{n_k+1}. \tag{7}$$

Proof 2.2 *The inner sum in (5), in its Bernoulli symbols version,*

$$\sum_{l_1,\dots,l_r}\binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\dots\binom{k_r}{l_r}\mathcal{B}^{l_1}\dots\mathcal{B}^{l_r},$$

can be summed to

$$(1+\mathcal{B}_1)^{\bar{n}+r-\bar{k}}(1+\mathcal{B}_2)^{k_2}\dots(1+\mathcal{B}_r)^{k_r}.$$

The classical identity [2] *for Bernoulli symbols $\mathcal{B}+1=-\mathcal{B}$, with $\bar{n}$ defined in (6) reduces this to*

$$(-1)^{\bar{n}+1}\mathcal{B}_1^{\bar{n}+r-\bar{k}}\mathcal{B}_2^{k_2}\dots\mathcal{B}_r^{k_r}. \tag{8}$$

It follows that

$$\zeta_r(-\mathbf{n}) = \frac{(-1)^{\bar{n}}}{(n_r+1)}\sum_{k_2,\dots,k_r}\mathcal{C}_1^{\bar{n}+r-\bar{k}}\mathcal{B}_2^{k_2}\dots\mathcal{B}_r^{k_r}\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}.$$

Summing first over k_2 gives

$$\zeta_r(-\mathbf{n}) = \frac{(-1)^{\bar{n}}}{(n_r+1)}\sum_{k_3,\dots,k_r}\mathcal{C}_1^{n_1+1}\mathcal{C}_2^{n_2+\dots+n_r+r-1}\mathcal{B}_3^{k_3}\dots\mathcal{B}_r^{k_r}\prod_{j=3}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}.$$

The result now follows by summing, in order, over the remaining indices.

Observe that the reduction (8) performed in the proof allows to restate a simpler version of Sadaoui's formula (5) as the more tractable $(r-1)$−fold sum

$$\zeta_r(-n_1,\dots,-n_r) = (-1)^{\bar{n}}\sum_{k_2,\dots,k_r}\frac{1}{(\bar{n}+r-\bar{k})}\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}B_{l_1}\dots B_{l_r}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}. \tag{9}$$

Observe moreover that the derivation of (7) is unchanged if the symbols $\mathcal{B}_1,\dots,\mathcal{B}_r$ are replaced by a generalization of the Bernoulli symbol $\mathcal{B}$, namely the polynomial Bernoulli symbol $\mathcal{B}+z$ defined by

$$(\mathcal{B}+z)^n = B_n(z),$$

2. this identity can be deduced from the generating function

$$\exp(z\mathcal{B}) = \frac{z}{\exp(z)-1}.$$

the Bernoulli polynomial of degree n. The same proof as above yields the next statement.

Theorem 2.3 *The analytic continuation of the zeta function as given in [1] can be written as*

$$\zeta_r(-n_1,\dots,-n_r,z_1,\dots,z_r)=\prod_{i=1}^{r}\mathcal{C}_{1,\dots,i}^{n_i+1}(z_1,\dots,z_i) \tag{10}$$

with

$$\mathcal{C}_1^n(z_1)=\frac{(z_1+\mathcal{B}_1)^n}{n}=\frac{B_n(z_1)}{n},\ \mathcal{C}_{1,2}^n(z_1,z_2)=\frac{(\mathcal{C}_1(z_1)+\mathcal{B}_2+z_2)^n}{n},\dots$$

and

$$\mathcal{C}_{1,2,\dots,k+1}^n(z_1,\dots,z_{k+1})=\frac{(\mathcal{C}_{1,2,\dots,k}(z_1,\dots,z_k)+\mathcal{B}_{k+1}+z_{k+1})^n}{n}.$$

3. A general recursion formula on the depth

The methods above are now used to produce a general recursion formula on the depth of the zeta function.

Theorem 3.1 *The multiple zeta functions satisfy the recursion rule*

$$\zeta_r(-\mathbf{n};\mathbf{z})=\frac{(-1)^{n_r}}{n_r+1}\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}(-1)^k\zeta_{r-1}(-n_1,\dots,-n_{r-1}-k;\mathbf{z})B_{n_r+1-k}(z_r). \tag{11}$$

Introducing the new zeta symbol $\mathcal{Z}_r$ with the evaluation rule [3]

$$\mathcal{Z}_r^k=\zeta_r(-n_1,\dots,-n_{r-1},-n_r-k;\mathbf{z}),$$

this recursion rule can be written symbolically as

$$\zeta_r(-\mathbf{n};\mathbf{z})=(-1)^{n_r}\frac{(\mathcal{B}-\mathcal{Z}_{r-1})^{n_r+1}}{n_r+1}=\zeta_1(-n_r;-\mathcal{Z}_{r-1}). \tag{12}$$

Proof 3.2 *Start from (10) and expand the last term*

$$\mathcal{C}_{1,\dots,r}^{n_r+1}(z_1,\dots,z_r)=\frac{\left(\mathcal{C}_{1,\dots,r-1}^{n_{r-1}+1}(z_1,\dots,z_{r-1})+\mathcal{B}_r(z_r)\right)^{n_r+1}}{n_r+1}$$

by using the binomial formula to produce

$$\zeta_r(-n_1,\dots,-n_r,z_1,\dots,z_r)=\frac{(-1)^{n_r}}{n_{r+1}}\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}\left(\prod_{i=1}^{r-2}\mathcal{C}_{1,\dots,i}^{n_i+1}(z_1,\dots,z_i)\right)$$
$$\times\,\mathcal{C}_{1,\dots,r-1}^{n_r+1+k}(z_1,\dots,z_{r-1})\,\mathcal{B}_r^{n_r+1-k}(z_r).$$

Then identify

$$\left(\prod_{i=1}^{r-2}\mathcal{C}_{1,\dots,i}^{n_i+1}(z_1,\dots,z_i)\right)\mathcal{C}_{1,\dots,r-1}^{n_r+1+k}(z_1,\dots,z_{r-1})$$

as

$$(-1)^{n_1+\cdots+n_{r-2}+n_{r-1}+k}\zeta_{r-1}(-n_1,\dots,-n_{r-2},-n_{r-1}-k;\mathbf{z})$$

to obtain the desired result.

3. note that $\mathcal{Z}_r^0\neq 1$

Using the symbol $\mathcal{Z}$, this identity can be rewritten as

$$\zeta_r\left(-n_1,\ldots,-n_r,z_1,\ldots,z_r\right)=\frac{(-1)^{n_r}}{n_{r+1}}\left(\mathcal{B}-\mathcal{Z}_{r-1}\right)^{n_r+1}$$

and the initial value

$$\zeta_1\left(-n;z\right)=(-1)^n\,\frac{(z+\mathcal{B})^{n+1}}{n+1}$$

provides the stated recursion.

4. Contiguity identities

The multiple zeta function at negative integer values satisfies contiguity identities in the z variables. Two of them are presented here.

Theorem 4.1 *The zeta function satisfies the contiguity identity*

$$\zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r+1\right)=\zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r\right)+(-1)^{n_r}\left(z_r-\mathcal{Z}_{r-1}\right)^{n_r}.$$

Example 3. In the case of the zeta function of depth 2,

$$\zeta_2\left(-n_1,-n_2,z_1,z_2+1\right)=\zeta_2\left(-n_1,-n_2,z_1,z_2\right)+(-1)^{n_1+1}\left(z_2-\mathcal{Z}_1\right)^{n_2}$$

and the second term is expanded as

$$(-1)^{n_1+1}\sum_{k=0}^{n_2}\binom{n_2}{k}z_2^{n_2-k}\,(-1)^k\,\zeta_1\left(-n_1-k;z_1\right).$$

Proof 4.2 *Expand*

$$\zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r+1\right)=\frac{(-1)^{\bar{n}}}{n_r+1}\mathcal{C}_1^{n_1+1}\left(z_1\right)\ldots\mathcal{C}_{1,\ldots,r-2}^{n_{r-2}+1}\left(z_1,\ldots,z_{r-2}\right)$$
$$\times\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}\mathcal{C}_{1,\ldots,r-1}^{n_{r-1}+1+k}\left(z_1,\ldots,z_{r-1}\right)B_{n_r+1-k}\left(z_r+1\right).$$

and use the identity on Bernoulli polynomials

$$B_{n_r+1-k}\left(z_r+1\right)=B_{n_r+1-k}\left(z_r\right)+\left(n_r-k+1\right)z_r^{n_r-k}$$

to produce the result.

The corresponding result for a shift in the first variable admits a similar proof.

Theorem 4.3 *The depth-2 zeta function satisfies the contiguity identities*

$$\zeta_2\left(-n_1,-n_2,z_1+1,z_2\right)=\zeta_2\left(-n_1,-n_2,z_1,z_2\right)+\frac{(-1)^{n_1+n_2}}{n_2+1}z_1^{n_1}B_{n_2+1}\left(z_1+z_2\right).$$

5. A Generating Function

The generating function of the zeta values at negative integers is defined by

$$F_r\left(w_1,\dots,w_r\right)=\sum_{n_1,\dots,n_r\geq 0}\frac{w_1^{n_1}\dots w_r^{n_r}}{n_1!\dots n_r!}\zeta_r\left(-n_1,\dots,-n_r\right). \qquad (13)$$

A recurrence for F_r is presented below. The initial condition is given in terms of the generating function for Bernoulli numbers

$$F_B\left(w\right)=\sum_{n=0}^{+\infty}\frac{B_n}{n!}z^n=\frac{w}{e^w-1}.$$

Theorem 5.1 *The generating function of the zeta values at negative integers satisfies the recurrence*

$$F_r\left(w_1,\dots,w_r\right)=\frac{1}{w_r}\left[F_{r-1}\left(w_1,\dots,w_{r-1}\right)-F_B\left(-w_r\right)F_{r-1}\left(w_1,\dots,w_{r-2},w_{r-1}+w_r\right)\right]$$

with the initial value

$$F_1\left(w_1\right)=-\frac{1}{w_1}\left[e^{-w_1\mathcal{B}_1}-1\right]=\frac{1-F_B\left(-w_1\right)}{w_1}.$$

Moreover, the representation of the shift operator as $\exp\left(a\frac{\partial}{\partial w}\right)\circ f\left(w\right)=f\left(w+a\right)$ *and* $F_1\left(w,z\right)=-\frac{1}{w}\left[e^{-w\left(\mathcal{B}+z\right)}-1\right]$ *give the recursion symbolically as*

$$F_r\left(w_1,\dots,w_r\right)=F_1\left(w_r,-\frac{\partial}{\partial w_{r-1}}\right)\circ F_{r-1}\left(w_1,\dots,w_{r-1}\right),$$

so that

$$F_r\left(w_1,\dots,w_r\right)=F_1\left(w_r,-\frac{\partial}{\partial w_{r-1}}\right)\circ F_1\left(w_{r-1},-\frac{\partial}{\partial w_{r-2}}\right)\circ\dots\circ F_1\left(w_2,-\frac{\partial}{\partial w_1}\right)\circ F_1\left(w_1\right).$$

Proof 5.2 *Start from*

$$F_r\left(w_1,\dots,w_r\right)=\sum_{n_1,\dots n_r}\frac{w_1^{n_1}\cdots w_r^{n_r}}{n_1!\cdots n_r!}\left(-1\right)^{n_1+\dots+n_r}\mathcal{C}_1^{n_1+1}\cdots\mathcal{C}_{1,\dots,r}^{n_r+1}=\prod_{j=1}^{r}\mathcal{C}_{1,\dots,j}e^{-w_j\mathcal{C}_{1,\dots,j}},$$

and expand

$$\mathcal{C}_{1,\dots,r}e^{-w_r\mathcal{C}_{1,\dots,r}}=\sum_{n=0}^{\infty}\frac{\left(-w_r\right)^n}{n!}\cdot\frac{\left(-1\right)^{n+1}}{n+1}\left(\mathcal{C}_{1,\dots,r-1}+\mathcal{B}_r\right)^{n+1}=-\frac{1}{w_r}\left(e^{-w_r\left(\mathcal{C}_{1,\dots,r-1}+\mathcal{B}_r\right)}-1\right),$$

to deduce that $F_r\left(w_1,\dots,w_r\right)$ *is*

$$\frac{1}{w_r}\left(\prod_{j=1}^{r-1}\mathcal{C}_{1,\dots,j}e^{-w_j\mathcal{C}_{1,\dots,j}}\right)-\frac{1}{w_r}\left(\prod_{j=1}^{r-2}\mathcal{C}_{1,\dots,j}e^{-w_j\mathcal{C}_{1,\dots,j}}\right)e^{-w_r\mathcal{B}_r}\mathcal{C}_{1,\dots,r-1}e^{-\left(w_{r-1}+w_r\right)\mathcal{C}_{1,\dots,r-1}}$$
$$=\frac{1}{w_r}F_{r-1}\left(w_1,\dots,w_{r-1}\right)-\frac{1}{w_r}F_B\left(-w_r\right)F_{r-1}\left(w_1,\dots,w_{r-2},w_{r-1}+w_r\right).$$

6. Shuffle Identity

Multiple zeta values at positive integers satisfy *shuffle identities*, such as

$$\zeta_2(n_1,n_2)+\zeta_2(n_2,n_1)+\zeta_1(n_1+n_2)=\zeta_1(n_1)\,\zeta_1(n_2)\,.$$

The analytic continuation technique used in [1] does not preserve this identity at negative integers, while others do (for example, see [6]). The following theorem gives the correction terms.

Theorem 6.1 *The zeta values at negative integers as defined in [1] satisfy the identity*

$$\zeta_2(-n_1,-n_2)+\zeta_2(-n_2,-n_1)+\zeta_1(-n_1-n_2)-\zeta_1(-n_1)\,\zeta_1(-n_2)=\frac{(-1)^{n_1+1}n_1!n_2!}{(n_1+n_2+2)!}B_{n_1+n_2+2}. \quad (14)$$

Remark 1. When n_1+n_2 is odd, $B_{n_1+n_2+2}=0$ so that the shuffle identity (14) holds for $\zeta_2(-n_1,-n_2)$ as expected, since the depth-2 zeta function is holomorphic at these points.

Proof 6.2 *Let* $\delta(w_1,w_2)=F_2(w_1,w_2)+F_2(w_2,w_1)+F_1(w_1+w_2)-F_1(w_1)\,F_1(w_2)\,.$ *An elementary calculation gives*

$$\delta(w_1,w_2)=\frac{\frac{1}{w_1}+\frac{1}{w_2}-\frac{1}{2}\coth\left(\frac{w_1}{2}\right)-\frac{1}{2}\coth\left(\frac{w_2}{2}\right)}{w_1+w_2}.$$

The expansions

$$\frac{1}{w_1}-\frac{1}{2}\coth\left(\frac{w_1}{2}\right)=-\sum_{k=0}^{+\infty}\frac{w_1^{2k+1}}{(2k+2)!}B_{2k+2} \text{ and } \frac{1}{w_1+w_2}=\frac{1}{w_2}\sum_{l\geq 0}\left(-\frac{w_1}{w_2}\right)$$

now produce

$$\delta(w_1,w_2)=-\sum_{k,l=0}^{+\infty}(-1)^l\,\frac{B_{2k+2}}{(2k+2)!}\left(w_1^{2k+l+1}w_2^{-l-1}+w_1^l w_2^{2k-l}\right).$$

Identifying the coefficient of $w_1^{n_1}w_2^{n_2}$ *in this series expansion gives the result.*

7. Specific multiple zeta values

This final section gives some examples of the evaluation at negative integers of the zeta function, obtained from (5) and (12).

1: for depth $r=2$,

$$\zeta_2(-n,0)=(-1)^n\left[\frac{B_{n+2}}{n+2}-\frac{1}{2}\frac{B_{n+1}}{n+1}\right], \quad (15)$$

and

$$\zeta_2(0,-n)=\frac{(-1)^{n+1}}{n+1}\left[B_{n+1}+B_{n+2}\right].; \quad (16)$$

2: for depth $r=3$,

$$\zeta_3(-n,0,0)=\frac{(-1)^n}{2}\left[\frac{B_{n+3}}{n+3}-2\frac{B_{n+2}}{n+2}+\frac{2}{3}\frac{B_{n+1}}{n+1}\right] \quad (17)$$

and

$$\zeta_3(0,-n,0)=\frac{(-1)^{n+1}}{2}\left[\frac{n}{(n+1)(n+2)}B_{n+2}-\frac{B_{n+1}}{n+1}+2\frac{B_{n+3}}{n+2}\right]. \tag{18}$$

3: as a final example, the recursion rule (11) is used to compute the value $\zeta_3(0,0,-2)$ as

$$\begin{aligned}\zeta_3(0,0,-2)&=\frac{(\mathcal{B}-\mathcal{Z}_2)^3}{3}=\frac{1}{3}\left(\mathcal{B}^3\mathcal{Z}_2^0-3\mathcal{B}^2\mathcal{Z}_2^1+3\mathcal{B}\mathcal{Z}_2^2-\mathcal{Z}_2^3\right)\\&=\frac{1}{3}\left(B_3\zeta_2(0,0)-3B_2\zeta_2(0,-1)+3B_1\zeta_2(0,-2)-\zeta_2(0,-3)\right)=-\frac{1}{60}.\end{aligned}$$

Acknowledgements

The first author acknowledges the partial support of NSF-DMS 0713836. The second author is partially supported, as a graduate student, by the same grant. The work of the third author was partially funded by the iCODE Institute, a research project of the Idex Paris-Saclay.

References

[1] B. Sadaoui, Multiple zeta values at the non-positive integers, C. R. Acad. Sci. Paris, Ser. 1, 2014, 12-352, 977-984

[2] D. Zagier, Values of zeta functions and their applications, Progress in Math., 120, 1994, 497–512

[3] S.Akiyama S.Egami and Y.Tanigawa, An analytic continuation of multiple zeta functions and their values at non-positive integers," Acta Arith. XCVIII, 2001, 107–116

[4] K. Matsumoto, The analytic continuation and the asymptotic behavior of certain multiple zeta-functions I, J. Number Theory 101, 2003, 223–243

[5] D. J. Broadhurst, Exploiting the 1,440-fold symmetry of the master two-loop diagram, Zeit. Phys. C 32, 1986, 249–253

[6] D. Manchon and S. Paycha, Nested sums of symbols and renormalised multiple zeta functions, preprint, arXiv:math/0702135

[7] T. Takamuki, The Kontsevich invariant and relations of multiple zeta values, Kobe J. Math. 16, 1999, 27–43

[8] Zhao, J.: Analytic continuation of multiple zeta-functions. Proc. Am. Math. Soc. 128, 2000, 1275–1283