A SEQUENCE OF UNIMODAL POLYNOMIALS

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Abstract. The purpose of this paper is to establish the unimodality and to determine the mode of a class of Jacobi polynomials which arises in the exact integration of certain rational functions as well as in the Taylor expansion of the double square root.

1. Introduction

A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *unimodal* if there exists an index $0 \leq j \leq m$ such that $d_0 \leq d_1 \leq \cdots \leq d_j$ and $d_j \geq d_{j+1} \geq \cdots \geq d_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal. The sequence $\{d_0, d_1, \cdots, d_m\}$ with $d_j \geq 0$ is said to be *logarithmically concave* (or *log concave* for short) if $d_{j+1}d_{j-1} \leq d_j^2$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is log concave then it is unimodal [16].

Unimodal polynomials arise often in combinatorics, geometry, and algebra, and have been the subject of considerable research. The reader is referred to [11, 6] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal. In this paper we prove the unimodalityof a specific class of Jacobi polynomials. The general Jacobi polynomials $P_m^{(\alpha,\beta)}(z)$ can be defined by

$$P_m^{(\alpha,\beta)}(z) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{z+1}{2} \right)^k \tag{1.1}$$

(see [1], page 189) or by

$$P_m^{(\alpha,\beta)}(z) = \frac{(-m-\beta)_m}{m!} \, {}_2F_1 \left[-m, m+1+\alpha+\beta, 1+\beta, \frac{1+z}{2} \right].$$

([8], 8.962.1). Here

$$ {}_2F_1[a,b,c;z] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k \tag{1.2}$$

is the hypergeometric function and $(r)_k$ is the rising factorial

$$(r)_k = r(r+1)(r+2)\cdots(r+k-1) = \frac{\Gamma(r+k)}{\Gamma(r)}.$$

Many classical families of polynomials are special cases of Jacobi polynomials. For instance the *Legendre polynomials* are $P_m^{(0,0)}(z)$ and the *Gegenbauer polynomials* are scalar multiples of $P_m^{(\lambda-1/2,\lambda-1/2)}(z)$. General information about these polynomials can be obtained in [13, 15].

Date: June 1, 1998.
Key words and phrases. Integrals, polynomials, unimodality, Wilf snake oil method.

We consider the polynomials

$$P_m(a) \quad = \quad \sum_{l=0}^{m} d_l(m) a^l, \tag{1.3}$$

with

$$d_l(m) \quad = \quad 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}. \tag{1.4}$$

The polynomials $P_m(a)$ arise in our development of a new procedure for the exact integration of rational functions, wherein we consider

$$N_{0,4}(a;m) \quad := \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.5}$$

We have shown [4]

$$N_{0,4}[a;m] \quad = \quad \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a) \tag{1.6}$$

where

$$P_m(a) \quad := \quad P_m^{(m+1/2,-m-1/2)}(a), \tag{1.7}$$

so that $P_m(a)$ is of the type $P_m^{(\alpha,\beta)}(a)$ with $\alpha = m + \frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$. After some simplification (1.7) yields

$$P_m(a) \quad = \quad 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k, \tag{1.8}$$

and expanding the powers of $a+1$ gives (1.4). We thus obtain

$$\begin{aligned} N_{0,4}(a;m) \quad &:= \quad \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \\ &= \quad \frac{\pi}{2^{3m+3/2}(a+1)^{m+1/2}} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k. \end{aligned}$$

This formula gives an efficient procedure for the evaluation of $N_{0,4}[a;m]$. For example

$$\frac{1}{\pi} \int_0^\infty \frac{dx}{(x^4+7x^2+1)^{50}} \quad =$$

$$\frac{114845664537973139383732728695907522557104064529084303055385344747186648 75}{756155814236193178352650772173678033029101516751105175397074035149880950784}.$$

Apart from its intrinsic interest, the sequence $N_{0,4}(a;m)$ appears as the coefficients of the Taylor expansion of the double square root:

$$\begin{aligned} y := \sqrt{a+\sqrt{1+c}} \quad &= \quad \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1) c^k \\ &= \quad \sqrt{a+1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} P_{k-1}(a)}{k 2^{k+1} (a+1)^k} c^k \right). \end{aligned} \tag{1.9}$$

(See [4] for details.) The power series expansion of roots of $\alpha q y^p - y^q + 1 = 0$ was initiated by Lagrange [9]. Examples of his technique, the Lagrange inversion formula, can be found in [7, 10]. In this case y, defined by (1.9) satisfies an algebraic equation from which a Lagrange-type expansion can be obtained; our expansion in (1.9) is simpler.

Two special cases of (1.9) appear in the literature. The case $a = 1$ appears in [7],

$$\sqrt{1+\sqrt{1+c}} \quad = \quad \sqrt{2}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{2^{4k-1}}\binom{4k-3}{2k}c^k\right),$$

and the case $c = a^2$,

$$\sqrt{a+\sqrt{1+a^2}} \quad = \quad 1+\frac{1}{2}a+\sum_{k=2}^{\infty}\frac{b_k(1/2)a^k}{k!}$$

where, for $k \geq 2$,

$$b_k(n) \quad = \quad \begin{cases} n^2(n^2-2^2)(n^2-4^2)\cdots(n^2-(k-2)^2), \text{ if } k \text{ is even,} \\ n(n^2-1^2)(n^2-3^2)\cdots(n^2-(k-2)^2), \text{ if } k \text{ is odd.}\end{cases}$$

is a special case of Corollary 2 to Entry 14 of Ramanujan's Notebooks, as described in [2].

A sufficient condition for unimodality of a polynomial is to have all its zeros real and negative(see [16] for a proof). This can be used to prove unimodality of a given sequence. For example, the signless Stirling numbers of the first kind, $\left[{n \atop k}\right]$, defined by their generating function

$$\sum_j \left[{n \atop k}\right] x^{j-1} \quad = \quad (x+1)(x+2)\cdots(x+n-1), \tag{1.10}$$

are unimodal.

In Section 3 we discuss the sequence of zeros of the polynomial $P_m(a)$. We show that all the zeros satisfy $|a+1| < 1$ and that $P_m(a)$ has the minimal number of real zeros that is possible: none for m even and 1 for m odd. We conjecture that, for m odd, the distance of the zeros to -1 is bounded from below by the modulus of the unique real zero. Our numerical studies suggest that the behavior of these zeros is analogous to that of the zeros of the partial sums of the exponential as discussed in [14].

2. Unimodality of the polynomial $P_m(a)$

In this section we prove that the coefficients of the polynomial P_m are unimodal. More precisely, we prove that the coefficients increase up to the central coefficient (i.e., the coefficient of $a^{[m/2]}$, and they decrease from then on. The proof is elementary in the sense that no property of the Jacobi family is employed.

Start with the expression (1.4) and define the difference

$$\Delta d_l(m) \quad = \quad d_{l+1}(m) - d_l(m). \tag{2.1}$$

We claim

Theorem 2.1. For fixed m, the polynomial $P_m(a)$ is unimodal. More precisely:
a) $\Delta\, d_l(m) > 0$ for $0 \leq l < \left[\frac{m}{2}\right]$
b) $\Delta\, d_l(m) < 0$ for $\left[\frac{m}{2}\right] \leq l \leq m-1$.

c) The smallest coefficient is the leading term

$$d_m(m) \quad = \quad 2^{-m}\binom{2m}{m}. \tag{2.2}$$

Part (c) follows immediately from parts (a) and (b). The remainder of this proof is divided into a sequence of lemmas.

Lemma 2.2. The difference $\Delta\, d_l(m)$ is given by

$$\Delta\, d_l(m) \quad = \quad \frac{1}{4^m}\binom{m+l}{m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m+l} \times \frac{k-2l-1}{l+1}. \tag{2.3}$$

Proof. This is elementary. □

Lemma 2.3. $\Delta\, d_l(m) < 0$ for $\left[\frac{m}{2}\right] \leq l \leq m-1$.

Proof. This follows directly from (2.3). If $l \leq k \leq m$ then

$$k - 2l - 1 \leq k - 2\left[\frac{m}{2}\right] - 1 \leq k - m \leq 0$$

and $k = l$ produces a strictly negative term. This proves part b). □

The proof of part a) is more delicate. First observe that the terms in (2.3) are positive for $k > 2l+1$ and negative otherwise. Therefore we need to prove

$$\sum_{k=l}^{2l} 2^k (2l+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+l} \quad < \quad \sum_{k=2l+2}^{m} 2^k (k-2l-1)\binom{2m-2k}{m-k}\binom{m+k}{m+l}. \tag{2.4}$$

Lemma 2.4. Let $0 \leq l < \left[\frac{m}{2}\right]$. Suppose

$$\sum_{k=l}^{2l} 2^k (2l+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+l} \quad < \quad \sum_{k=2l+2}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m+l} \tag{2.5}$$

holds. Then $\Delta\, d_l(m) > 0$ and the proof of part a) is finished.

Proof. This is obtained by replacing the term $k - 2l - 1$ on the right hand side of (2.4) by 1, and observe that this makes the required inequality stricter. □

Lemma 2.5. Let $0 \leq l < \left[\frac{m}{2}\right]$. Suppose

$$\sum_{k=l}^{2l} 2^k (2l+1-k)\binom{2m-2k}{m-k}\binom{m+k}{m+l} < 2^m \binom{2m}{m+l} \tag{2.6}$$

holds. Then $\Delta\, d_l(m) > 0$ and the proof of part a) is finished.

Proof. The inequality (2.5) is strengthen one more time by replacing the sum on the right by its last term. The lemma is proved. □

The inequality in (2.6) that is required to finish the proof, is now written as

$$S_{m,l} := \sum_{k=l}^{2l} \binom{m-l}{m-k}\binom{m+k}{2k}\binom{2m}{2k}^{-1} \times \frac{2l+1-k}{2^{m-k}} < 1. \tag{2.7}$$

Lemma 2.6. Suppose $S_{m,l} < 1$. Then $\Delta\, d_l(m) > 0$ for $0 \leq l < \left[\frac{m}{2}\right]$ and the proof of theorem 1 is complete.

We now study the sums $S_{m,l}$ and we first prove:

Lemma 2.7. For fixed m, the sum $S_{m,l}$ is *increasing* in l. In particular it is maximum when l is, i.e., at $l = \left[\frac{m-1}{2}\right]$. Therefore, if

$$S_{m,\left[\frac{m-1}{2}\right]} < 1 \tag{2.8}$$

then $\Delta\, d_l(m) > 0$ and the proof of Theorem 1 is finished.

Proof. The inequality $S_{m,l+1} > S_{m,l}$ is equivalent to

$$\begin{aligned} &\sum_{k=l+1}^{2l+2} \binom{m-l-1}{m-k}\binom{m+k}{2k}(2l+3-k)\binom{2m}{2k}^{-1} 2^{-m+k} \quad > \\ &\sum_{k=l}^{2l} \binom{m-l}{m-k}\binom{m+k}{2k}(2l+1-k)\binom{2m}{2k}^{-1} 2^{-m+k}. \end{aligned} \tag{2.9}$$

There are $l+2$ terms on the left term and $l+1$ on the right. To prove (2.9) it suffices to show that, for $j = 2l, 2l-1, \cdots, l$, the term corresponding to $k = j+2$ in $S_{m,l+1}$ is larger than the term corresponding to $k = j$ in $S_{m,l}$. This amounts to

$$\begin{aligned} &\binom{m-l-1}{m-j-2}\binom{m+j+2}{2j+4}(2l+1-j)2^{-m+j+2}\binom{2m}{2j+4}^{-1} \quad > \\ &\binom{m-l}{m-j}\binom{m+j}{2j}(2l+1-j)2^{-m+j}\binom{2m}{2j}^{-1}. \end{aligned} \tag{2.10}$$

Inequality (2.10) is equivalent to

$$X \;:=\; \frac{(m-j)(m-j-1)(m+j+2)(m+j+1)}{(m-l)(j-l+1)(2m-2j-1)(2m-2j-3)} > 1 \tag{2.11}$$

by direct computation.

We now show that $X > 1$. The proof is divided according to the parity of m. Suppose first that m is even, say $m = 2n$. Then $l \leq n-1$. We now show that the expression Y, obtained from X by replacing l by $n-1$ satisfies $Y > 1$. This finishes the proof, in view of $X \geq Y$. The expression Y is given by

$$Y \;=\; \frac{(2n-j)(2n-j-1)(2n+j+2)(2n+j+1)}{(n+1)(j-n+2)(4n-2j-1)(4n-2j-3)} \tag{2.12}$$

and it can be written as

$$Y = \frac{2(2n-j)}{4n-2j-1} \times \frac{2(2n-j-1)}{4n-2j-3} \times \frac{(n+1+j/2)}{n+1} \times \frac{(2n+j+1)/2}{j-n+2}. \tag{2.13}$$

The first three factors are clearly above 1. The last one is also bigger than 1 because $4n \geq 2j+4 > j+3$. This proves $X > 1$ for m even. The case m odd is handled in a similar form. The proof of the lemma is finished. □

We finally have:

Lemma 2.8. The maximal sum $S_{m,[\frac{m-1}{2}]}$ is strictly less than 1.

Proof. As before, the proof is divided according to the parity of m. We give the details for m even, say $m = 2n$. then $l = n-1$ and we need to show

$$\sum_{k=n-1}^{2n-2} \binom{4n-2k}{2n-k}\binom{n+1}{2n-k}(2n-1-k)2^{-2n+k}\binom{4n}{2n-k}^{-1} < 1. \tag{2.14}$$

The substitution $r = 2n-k$ show that we need to prove

$$\sum_{r=2}^{n+1} \binom{2r}{r}\binom{n+1}{r}(r-1)2^{-r}\binom{4n}{r}^{-1} < 1. \tag{2.15}$$

Define

$$a_{n,r} = \binom{2r}{r}\binom{n+1}{r}(r-1)2^{-r}\binom{4n}{r}^{-1}, \text{ for } 2 \leq r \leq n+1. \tag{2.16}$$

We now show that the lemma follows from the estimate

$$\frac{a_{n,r+1}}{a_{n,r}} < \frac{5}{6} \tag{2.17}$$

Proof. From (2.17) it follows that

$$a_{n,r} < a_{n,2} \times \left(\frac{5}{6}\right)^{r-2}$$

so

$$\begin{aligned} \sum_{r=2}^{n+1} a_{n,r} &< a_{n,2}\sum_{r=2}^{n+1}\left(\frac{5}{6}\right)^{r-2} \\ &< 6\left[1-\left(\frac{5}{6}\right)^{n}\right]a_{n,2} \end{aligned}$$

and using $a_{n,2} = 3(n+1)/[8(4n-1)]$, and the bound $(n+1)/(4n-1) \leq 3/7$ for $n \geq 2$, we get

$$\begin{aligned} \sum_{r=2}^{n+1} a_{n,r} &< \frac{27}{28}\left[1-\left(\frac{5}{6}\right)^{n}\right] \\ &< 1. \end{aligned}$$

The case $n = 1$ is simple: $a_{1,2} = 1/2$. □

Therefore, the proof of Theorem 1 is reduced to the proof of the estimate (2.17). □

Proof of (2.17). Define

$$q_{n,r} = \frac{a_{n,r+1}}{a_{n,r}} \text{ for } 2 \leq r \leq n$$

so that

$$q_{n,r} = \frac{r(2r+1)(n+1-r)}{(r+1)(r-1)(4n-r)}. \tag{2.18}$$

We first observe that $q_{n,r}$ is strictly increasing with n. Indeed,

$$\frac{q_{n+1,r}}{q_{n,r}} = \frac{(n-r+2)(4n-r)}{(n-r+1)(4n+4-r)}$$

and for $r=2$ we have

$$\frac{q_{n+1,2}}{q_{n,2}} = \frac{2n^2-n}{2n^2-n-1} > 1.$$

Now, for $r \geq 3$ we have

$$\begin{aligned}
\frac{q_{n+1,r}}{q_{n,r}} &= \left\{1+\frac{1}{n-r+1}\right\}\left\{1+\frac{4}{4n-r}\right\}^{-1} \\
&> \left\{1+\frac{1}{n-r+1}\right\}\left\{1-\frac{4}{4n-r}\right\} \\
&= 1+\frac{(3r-8)}{(n-r+1)(4n-r)} \\
&> 1.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in (2.18), with fixed r, we obtain

$$q_{n,r} < \lim_{n\to\infty} q_{n,r} = \frac{r(2r+1)}{4(r+1)(r-1)}$$

and an elementary calculation shows that the right hand side is decreasing for $r \geq 2$. We conclude that

$$\frac{a_{n,r+1}}{a_{n,r}} = q_{n,r} < \frac{2(2\cdot 2+1)}{4(2+1)(2-1)} = \frac{5}{6}.$$

The final statement c) is a consequence of the elementary inequality

$$\prod_{l=1}^{m}(m+l) < \prod_{l=1}^{m}(4l-1).$$

This completes the proof of the theorem.

There are other classes of Jacobi polynomials that are unimodal. We propose:

Problem 2.9. Let $m \in \mathbb{N}$ and $0 < j < 2n$. Then the sequence of polynomials

$$P_m^j(a) := P_m^{(m+1,-(2m-j))}(a)$$

is unimodal. The coefficients a_k increase from the constant a_0 to $a_{[\frac{j+1}{2}]}$ and decrease from then on.

Note. We have computed the sum (2.15) for large values of n, and we conjecture

$$\lim_{n\to\infty}\sum_{r=2}^{n+1}\binom{2r}{r}\binom{n+1}{r}(r-1)2^{-r}\binom{4n}{r}^{-1} = 1-\ln 2. \tag{2.19}$$

Using Mathematica 3.0 we found that

$$\begin{aligned} S(n) &:= \binom{2r}{r}\binom{n+1}{r}(r-1)2^{-r}\binom{4n}{r}^{-1} \\ &= 1-{}_2F_1\left[\frac{1}{2},-1-n,-4n;2\right]+\frac{1+n}{2}{}_2F_1\left[\frac{3}{2},-n,1-4n;2\right] \end{aligned}$$

so perhaps it is possible to prove (2.19) from here.

3. The structure of the zeros

In this section we discuss the location and nature of the zeros of the polynomial $P_m(a)$. We employ the explicit expression

$$P_m(a) = 2^{-2m}\sum_{k=0}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k \tag{3.1}$$

to obtain a bound on the zeros and then the fact that $P_m(a)$ is a Jacobi polynomial

$$P_m(a) = P_m^{(m+1/2,-m-1/2)}(a)$$

to determine the exact number and location of the *real zeros* of $P_m(a)$.

3.1. The real zeros.

We use the fact that $P_m(a) = P_m^{(m+1/2,-m-1/2)}(a)$ is part of the Jacobi family to determine the number of real zeros. This number of such zeros is obtained by a formula developed by Klein, Hilbert and Stieltjes. Introduce the Klein symbol E(u) via

$$E(u) = \begin{cases} 0 \text{ if } u\le 0 \\ [u] \text{ if } u>0 \text{ and } u \text{ is non-integral} \\ u-1 \text{ if } u=1,2,\cdots. \end{cases} \tag{3.1}$$

Denote by N_1, N_2, N_3 the number of zeros of $P_m^{(\alpha,\beta)}(a)$ in $-1<a<1,\ a<-1$, and $a>1$, respectively. Then the values N_i can be expressed in terms of

$$\begin{aligned} X &= X(\alpha,\beta) = E\left(\tfrac{1}{2}(|2m+\alpha+\beta+1|-|\alpha|-|\beta|+1)\right) \\ Y &= Y(\alpha,\beta) = E\left(\tfrac{1}{2}(-|2m+\alpha+\beta+1|+|\alpha|-|\beta|+1)\right) \\ Z &= Z(\alpha,\beta) = E\left(\tfrac{1}{2}(-|2m+\alpha+\beta+1|-|\alpha|+|\beta|+1)\right) \end{aligned}$$

via the formula

$$\begin{aligned} N_1 &= \begin{cases} 2\left[\frac{X+1}{2}\right] \text{ if } (-1)^m\binom{m+\alpha}{m}\binom{m+\beta}{m}>0 \\ 2\left[\frac{X}{2}\right]+1 \text{ if } (-1)^m\binom{m+\alpha}{m}\binom{m+\beta}{m}<0 \end{cases} \\ N_2 &= \begin{cases} 2\left[\frac{Y+1}{2}\right] \text{ if } \binom{2m+\alpha+\beta}{m}\binom{m+\beta}{m}>0 \\ 2\left[\frac{Y}{2}\right]+1 \text{ if } \binom{2m+\alpha+\beta}{m}\binom{m+\beta}{m}<0 \end{cases} \\ N_3 &= \begin{cases} 2\left[\frac{Z+1}{2}\right] \text{ if } \binom{2m+\alpha+\beta}{m}\binom{m+\alpha}{m}>0 \\ 2\left[\frac{Z}{2}\right]+1 \text{ if } \binom{2m+\alpha+\beta}{m}\binom{m+\alpha}{m}<0. \end{cases} \end{aligned}$$

This is formula is explained in Szego [13], page 145. In our case $\alpha = m + 1/2$ and $\beta = -(m+1/2)$ so that

$$\begin{aligned} X &= E\left\{\tfrac{1}{2}(|2m+1| - |m+1/2| - |m+1/2| + 1)\right\} = E\{\tfrac{1}{2}\} = 0 \\ Y &= E\left\{\tfrac{1}{2}(-|2m+1| + |m+1/2| - |m+1/2| + 1)\right\} = E\{-m\} = 0 \\ Z &= E\left\{\tfrac{1}{2}(-|2m+1| - |m+1/2| + |-m-1/2| + 1)\right\} = E\{-m\} = 0. \end{aligned}$$

We also have

$$\begin{aligned} \binom{m+\alpha}{m} &= \binom{2m+1/2}{m} > 0 \\ \binom{m+\beta}{m} &= \binom{-1/2}{m} = (-1)^m \times \frac{(1/2)(1/2+1)\cdots(1/2+m-1)}{m!} \\ &= (-1)^m \times \text{ a positive factor} \\ \binom{2m+\alpha+\beta}{m} &= \binom{2m}{m} > 0. \end{aligned}$$

From here it follows that

$$N_1 = 2[(X+1)/2] = 0$$

so there are no zeros for $-1 < a < 1$.

Similarly

$$N_2 = \begin{cases} 2[(Y+1)/2] = 0 \text{ if } (-1)^m > 0 \\ 2[Y/2] + 1 = 1 \text{ if } (-1)^m < 0 \end{cases}$$

so there are no zeros for $a < -1$ if m is even and a single real zero if m is odd. Finally

$$N_3 = 2[(Z+1)/2] = 0$$

so there are no zeros for $a > 1$. We have proven

Theorem 3.1. The polynomial $P_m(a)$ has no real zeros for m even and a single real zero, located in $a < -1$, for m odd.

3.2. **Bounds on the zeros and numerical calculations.** We now establish upper bounds for the modulus of the zeros of $P_m(a)$ and describe the results of their numerical calculations.

Define

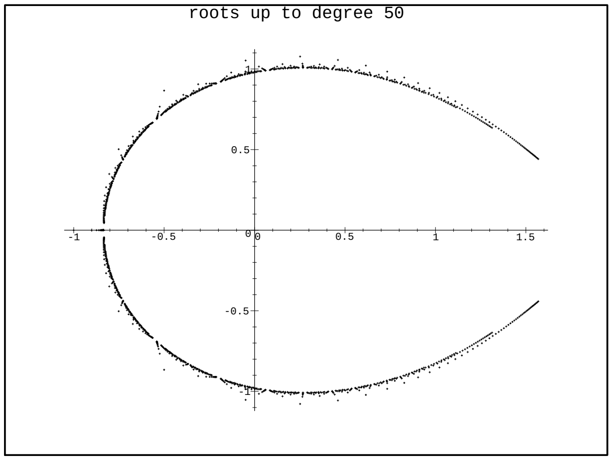
$$c_k(m) = 2^{-2m+k}\binom{2m-2k}{m-k}\binom{m+k}{m} \tag{3.1}$$

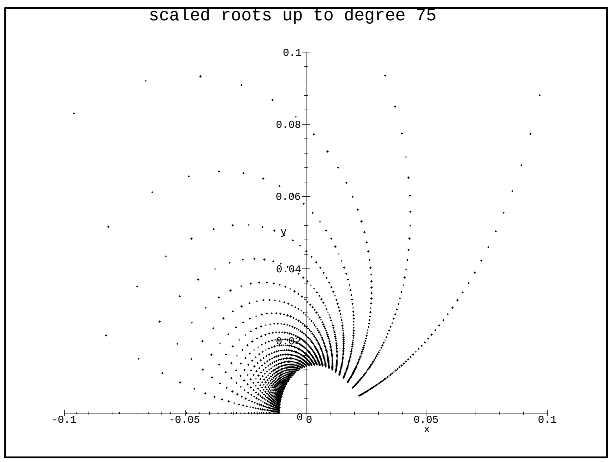
then

$$\frac{c_{k+1}}{c_k} = \frac{(m-k)(m+k+1)}{2m-2k-1)(k+1)} > 1, \text{ for } 0 \leq k \leq m-1.$$

Therefore the coefficients of P_m, as a polynomial in $b = a+1$, are positive and increasing. The Enerstrom-Kakeya theorem (see [5], page 12) guarantees that all its zeros are inside the unit circle of the b-plane:

Theorem 3.2. Let $a_j : 1 \leq j \leq m$ be the sequence of zeros of $P_m(a) = 0$. Then $|a_j + 1| < 1$.





We have computed numerical approximations to the zeros a_j. These calculation indicate that the bound in Theorem 3 is optimal: we propose

Problem 3.3. Prove that

$$\lim_{m \to \infty} \max\{|a_j + 1| : 1 \leq j \leq m\} \quad = \quad 1$$

In figure 1 we show the zeros of $P_{75}(a)$. The behavior is typical: the zeros are concentrated in a narrow *oval shape curve*. Moreover, for m odd, the zero of smallest modulus is the real zero $a_{real} < -1$.

Problem 3.4. Prove that

$$\min\{|a_j| : 1 \leq j \leq m\} \quad = \quad -a_{real}.$$

In figure 2 we show the zeros of all the polynomials in the sequence $P_m(a)$ for $1 \leq m \leq 75$. We observe that these zeros concentrate in a narrow *lemniscatic region*. A similar behavior is observed in the study of the zeros of partial sums $s_m(a)$ of the exponential function, see [14] for details. In this case, Szego [12] considered the normalized sequence $s_m(ma)$ and proved that the limit points of the zeros of the normalized polynomial fill the part of the lemniscate $|ae^{1-a}| = 1$ inside the closed unit circle. In our case, the zeros of the normalized sequence $P_m(ma)$ converge to 0. We have observed that as they do, they form an inner lemniscate, but we have been unable to predict its equation.

4. Conclusions

In this paper we have shown that the coefficients of the Jacobi polynomial

$$P_m(a) := P_m^{(m+1/2,-m-1/2)}(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k$$

are unimodal. We have also examined the structure of the zeros of $P_m(a)$.

The refernces also include [3].

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