THE ZAGIER MODIFICATION OF BERNOULLI NUMBERS AND A POLYNOMIAL EXTENSION. PART I.

ATUL DIXIT, VICTOR H. MOLL, AND CHRISTOPHE VIGNAT

Abstract. The modified Bernoulli numbers

$$B_n^* = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r}, \quad n > 0$$

introduced by D. Zagier in 1998 are extended to the polynomial case by replacing B_r by the Bernoulli polynomials $B_r(x)$. Properties of these new polynomials are established using the umbral method as well as classical techniques. The values of x that yield periodic subsequences $B_{2n+1}^*(x)$ are classified. The strange 6-periodicity of B_{2n+1}^*, established by Zagier, is explained by exhibiting a decomposition of this sequence as the sum of two parts with periods 2 and 3, respectively. Similar results for modifications of Euler numbers are stated.

1. Introduction

The Bernoulli numbers B_n, defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1.1}$$

are rational numbers with $B_{2n+1} = 0$ for $n \geq 1$ and $B_1 = -\frac{1}{2}$. The sequence $\{B_n\}$ has remarkable properties and it appears in a variety of mathematical problems. Examples of such include the fact that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.2}$$

evaluated at an even positive integer $s = 2n$ is a rational multiple of π^{2n}, the factor being

$$\frac{\zeta(2n)}{\pi^{2n}} = \frac{2^{2n-1}}{(2n)!}(-1)^{n-1} B_{2n}. \tag{1.3}$$

Their denominators are completely determined by the von Staudt-Clausen theorem: the denominator of B_{2n} is the product of all primes p such that $p-1$ divides $2n$ (see [11] for an elementary proof). It is often the numerators of B_{2n} that are the objects of interest. It is a remarkable mystery that there is no elementary formula associated to them. These numerators appear in connection to Fermat's last theorem (see [14]) and also in relation to the group of smooth structures on n-spheres (see [9], page 530 and [10] for details).

Date: September 18, 2012.

1991 *Mathematics Subject Classification.* Primary 11B68, 33C45.

Key words and phrases. Bernoulli polynomials, Chebyshev polynomials, umbral method, periodic sequences, Euler polynomials, generating functions, WZ-method.

D. Zagier [21] introduced the modified Bernoulli numbers

$$B_n^* = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r}, \quad n > 0 \tag{1.4}$$

and established the following *amusing variant* of $B_{2n+1} = 0$:

Theorem 1.1. *The sequence B_{2n+1}^* is periodic of period 6 with values*

$$\{\tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{3}{4}\}.$$

One of the goals of this work is to extend this result to the polynomial

$$B_n^*(x) = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r(x)}{n+r}, \quad n > 0 \tag{1.5}$$

which is defined here as the *Zagier polynomial*. Here $B_r(x)$ is the classical Bernoulli polynomial defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.6}$$

The objective of the paper is to produce analogues of standard results for $B_n(x)$ for the Zagier polynomials $B_n^*(x)$. For example, a generating function for these polynomials appears in Theorem 5.1 as

$$\sum_{n=1}^{\infty} B_n^*(x) z^n = -\frac{1}{2} \log z - \frac{1}{2} \psi\left(z + 1/z - 1 - x\right) \tag{1.7}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. The generating function (1.7) really corresponds to the less elementary expression

$$\sum_{n=0}^{\infty} B_n(x) z^n = \frac{1}{z} \zeta(2, 1/z - x + 1). \tag{1.8}$$

Here $\zeta(s, a)$ is the Hurwitz zeta function, defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}. \tag{1.9}$$

A derivation of (1.8) is given in Section 5. The next example corresponds to the derivative rule $B_n'(x) = nB_{n-1}(x)$ for the Bernoulli polynomials. It appears in Theorem 8.2 as

$$\frac{d}{dx} B_n^*(x) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (2j-1) B_{2j-1}^*(x) \quad \text{for } n \text{ even}$$

and

$$\frac{d}{dx} B_n^*(x) = \frac{1}{2} + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} 2j B_{2j}^*(x) \quad \text{for } n \text{ odd}.$$

Finally, the analogue of the symmetry relation $B_n(1-x) = (-1)^n B_n(x)$ is established as

$$B_n^*(-x-3) = (-1)^n B_n^*(x). \tag{1.10}$$

This is the content of Theorem 11.1.

The original motivating factor for this work was to extend Theorem 1.1 to other values of $B^*_{2n+1}(x)$. The main result presented here is a classification of the values $x \in \mathbb{R}$ for which $B^*_{2n+1}(x)$ is a periodic sequence, Zagier's case being $x = 0$.

Theorem 1.2. *Assume $\{B^*_{2n+1}(x)\}$ is a periodic sequence. Then $x \in \{-3, -2, -1, 0\}$ or $x = -\frac{3}{2}$ where $B^*_{2n+1}\left(-\frac{3}{2}\right) = 0$.*

In the case of even degree, the natural result is expressed in terms of the difference $A^*_{2n}(x) = B^*_{2n}(x) - B^*_{2n}(-1)$.

Theorem 1.3. *Assume $\{A^*_{2n}(x)\}$ is a periodic sequence. Then $x \in \{-1, 0, 1, 2\}$. The period is 3 for $x = -1$ and $x = 2$ while $A^*_{2n}(x)$ vanishes identically for $x = 0$ and $x = 1$.*

An outline of the paper is given next. Section 2 contains a basic introduction to the umbral calculus with a special emphasis on the operational rules for the Bernoulli umbra. Section 3 gives the generating function of the modified Bernoulli numbers B^*_n and this is used to give a proof of the 6-periodicity of B^*_{2n+1}. An inversion formula expressing B_n in terms of B^*_n is given in Section 4. The proof extends to the polynomial case. The generating function for the Zagier polynomial $B^*_n(x)$ is established in Section 5 and an introduction to the arithmetic properties of special values of these polynomials appears in Section 6. Expressions for the derivatives of the Zagier polynomials are given in Section 8. Some binomials sums employed in the proof of these results are given in Section 7. The basic properties of Chebyshev polynomials are reviewed in Section 9 and used in Section 10 to give a representation of the Zagier polynomials in terms of Bernoulli and Chebyshev polynomials and also to prove a symmetry property of $B^*_n(x)$ in Section 11. Section 12 contains one of the main results: the classification of all periodic sequences of the form $B^*_{2n+1}(x)$. This result extends the original theorem of D. Zagier on the 6-periodicity of B^*_{2n+1}. Several additional properties of the Zagier polynomials are stated in Section 13. The results discussed in the present paper can be extended without difficulty to the case of Euler polynomials. These extensions are stated in Section 14 and used in Section 15 to establish a duplication formula for Zagier polynomials.

2. The umbral calculus

In the classical *umbral calculus*, as introduced by J. Blissard [1], the terms a_n of a sequence are formally replaced by the powers $\mathfrak{a}^n$ of a new variable $\mathfrak{a}$, named the *umbra* of the sequence $\{a_n\}$. The original sequence is recovered by the *evaluation map*

$$\text{eval}\,\{\mathfrak{a}^n\} = a_n. \tag{2.1}$$

The introduction of an umbra for $\{a_n\}$ requires a *constitutive equation* that reflects the properties of the original sequence. These ideas are illustrated with the umbra $\mathfrak{B}$ of the Bernoulli numbers $\{B_n\}$.

An alternative approach to (1.1), as a definition for the Bernoulli numbers B_n, is to use the equivalent recursion formula

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \qquad \text{for } n > 1, \tag{2.2}$$

complemented by the initial condition $B_0 = 1$. In terms of the Bernoulli umbra $\mathfrak{B}$, this recursion is written as

$$-\mathfrak{B} = \mathfrak{B} + 1. \tag{2.3}$$

This is a constitutive equation for the Bernoulli umbra and the numbers B_n are then obtained via the evaluation map

$$\text{eval}\{\mathfrak{B}^n\} = B_n. \tag{2.4}$$

The umbral method is illustrated by computing the first few values of B_n, starting with the initial condition $B_0 = 1$. The choice $n = 2$ in (2.3) gives

$$\mathfrak{B}^2 = (\mathfrak{B} + 1)^2 = \mathfrak{B}^2 + 2\mathfrak{B}^1 + \mathfrak{B}^0. \tag{2.5}$$

The evaluation map then gives $B_2 = B_2 + 2B_1 + B_0$ that simplifies to $2B_1 + B_0 = 0$ and this yields $B_1 = -1/2$. Similarly, $n = 3$ gives

$$\mathfrak{B}^3 = (\mathfrak{B} + 1)^3 = \mathfrak{B}^3 + 3\mathfrak{B}^2 + 3\mathfrak{B}^1 + \mathfrak{B}^0, \tag{2.6}$$

and the evaluation map produces $B_3 + 3B_2 + 3B_1 + B_0 = B_3$ and $B_2 = \frac{1}{6}$ is obtained. The reader will find more details about these ideas in [6].

The evaluation map of the Bernoulli umbra $\mathfrak{B}$ may be defined at the level of generating functions by

$$\text{eval}\left\{\exp(t\mathfrak{B})\right\} = \frac{t}{e^t - 1}. \tag{2.7}$$

Similarly, the extension of (2.7) to the umbrae $\mathfrak{B}(x)$ for the Bernoulli polynomials in (1.6) is defined by

$$\text{eval}\left\{\exp(t\mathfrak{B}(x))\right\} = \frac{te^{xt}}{e^t - 1}. \tag{2.8}$$

It is a general statement about umbral calculus that the operation *eval* is linear. Moreover, expressions independent of the corresponding umbra are to be treated as constant with respect to *eval*. Some further operational rules, particular for the Bernoulli umbra, are stated next.

Lemma 2.1. *The relation*

$$\text{eval}\{\mathfrak{B}(x)\} = \text{eval}\{x + \mathfrak{B}\}. \tag{2.9}$$

holds.

Proof. This comes directly from

$$\text{eval}\left\{\exp(t\mathfrak{B}(x))\right\} = \frac{te^{xt}}{e^t - 1} \text{ and } \text{eval}\left\{\exp(t\mathfrak{B})\right\} = \frac{t}{e^t - 1}. \tag{2.10}$$

□

Lemma 2.2. *Let P be a polynomial. Then*

$$\text{eval}\left\{P(x + \mathfrak{B} + 1)\right\} = \text{eval}\left\{P(x + \mathfrak{B})\right\} + P'(x). \tag{2.11}$$

Proof. This is verified first for monomials using (2.3) and then extended by linearity to the polynomial case. □

The next step is to present a procedure to evaluate nonlinear functions of the Bernoulli polynomials. This can be done using the umbral approach but we introduce here an equivalent probabilistic formalism that is easier to use in a variety of examples. These two approaches will be described further in [5] where some of the results presented in [20] will be established using this formalism.

The notation

$$\mathbb{E}\left[h(X)\right] = \int_{\mathbb{R}} h(x)\, f_X(x)\, dx \tag{2.12}$$

is used here for the expectation operator based on the random variable X with probability density f_X. The class of admissible functions h is restricted by the existence of the integral (2.12). The equation (2.13) shows that a probabilistic equivalent of the *eval* operator of umbral calculus is the expectation operator with respect to the probability distribution (2.14).

Theorem 2.3. *There exists a real valued random variable L_B with probability density $f_{L_B}(x)$ such that, for all admissible functions h,*

$$\text{eval}\{h(\mathfrak{B}(x))\} = \mathbb{E}\left[h(x - 1/2 + iL_B)\right] \tag{2.13}$$

where the expectation is defined in (2.12)*. The density of L_B is given by*

$$f_{L_B}(x) = \frac{\pi}{2}\text{sech}^2(\pi x), \qquad \textit{for } x \in \mathbb{R}. \tag{2.14}$$

Proof. Put $x = 0$ in the special case

$$\text{eval}\left\{\exp(t\mathfrak{B}(x))\right\} = \mathbb{E}\left[\exp\left(t(x - \tfrac{1}{2} + iL_B)\right)\right] \tag{2.15}$$

of (2.13) and use (2.8) to produce

$$\mathbb{E}\left[\exp(itL_B)\right] = \frac{t}{2}\text{csch}\left(\frac{t}{2}\right). \tag{2.16}$$

Let $f_{L_B}(x)$ be the density of L_B and write (2.16) as

$$\int_{-\infty}^{\infty} \cos(tu) f_{L_B}(u)\, du = \frac{t}{2}\text{csch}\left(\frac{t}{2}\right) \tag{2.17}$$

assuming the symmetry of L_B. The result now follows from entry 3.982.1 in [7]

$$\int_{-\infty}^{\infty} \text{sech}^2(au)\, \cos(tu)\, du = \frac{\pi t}{a^2}\text{csch}\left(\frac{\pi t}{2a}\right). \tag{2.18}$$

$\square$

Note 2.4. The integral representation of the Bernoulli polynomials

$$B_n(x) = \mathbb{E}\left[(x - \tfrac{1}{2} + iL_B)^n\right]. \tag{2.19}$$

is a special case of Theorem 2.3. The formula (2.19) is stated in non-probabilistic language as

$$B_n(x) = \frac{\pi}{2}\int_{-\infty}^{\infty} \left(x - \tfrac{1}{2} + it\right)^n \ \text{sech}^2(\pi t)\, dt. \tag{2.20}$$

To the best of our knowledge, this evaluation first appeared in [19]. The role played by $\text{sech}^2 x$ as a solitary wave for the Kortweg-de Vries equation has prompted the titles of the evaluations of (2.20) in [2, 8].

The next result uses Theorem 2.3 to evaluate a nonlinear function of the Bernoulli polynomials that will be needed later. More examples will appear in the companion paper [5].

Theorem 2.5. *Let* $\psi(x) = \Gamma'(x)/\Gamma(x)$ *be the digamma function. Then*

$$\text{eval}\left\{\log \mathfrak{B}(x)\right\} = \psi\left(\tfrac{1}{2} + \left|x - \tfrac{1}{2}\right|\right). \tag{2.21}$$

In particular, for $x \geq \frac{1}{2}$,

$$\text{eval}\left\{\log \mathfrak{B}(x)\right\} = \psi(x). \tag{2.22}$$

Proof. Theorem 2.3 with $h(x) = \log x$ gives

$$\text{eval}\left\{\log \mathfrak{B}(x)\right\} = \mathbb{E}\left[\log\left(x - \tfrac{1}{2} + iL_B\right)\right]. \tag{2.23}$$

The density f_{L_B} is an even function, therefore the random variables L_B and $-L_B$ have the same distribution. This gives

$$\begin{aligned}
\text{eval}\left\{\log \mathfrak{B}(x)\right\} &= \frac{1}{2}\mathbb{E}\left[\log\left((x - \tfrac{1}{2})^2 + L_B^2\right)\right] \\
&= \log\left(x - \tfrac{1}{2}\right) + \frac{1}{2}\mathbb{E}\left[\log\left(1 + \frac{L_B^2}{(x - \frac{1}{2})^2}\right)\right].
\end{aligned}$$

A linear scaling of entry 4.373.4 in [7] gives

$$\int_0^\infty \frac{\log(1 + bu^2)}{\sinh^2 cu}\, du = \frac{2}{c}\left[\log \frac{c}{\pi\sqrt{b}} - \frac{\pi\sqrt{b}}{2c} - \psi\left(\frac{c}{\pi\sqrt{b}}\right)\right], \tag{2.24}$$

for b, $c > 0$. Define

$$h(b, c) := \frac{2}{c}\left[\log \frac{c}{\pi\sqrt{b}} - \frac{\pi\sqrt{b}}{2c} - \psi\left(\frac{c}{\pi\sqrt{b}}\right)\right] \tag{2.25}$$

and observe that

$$\begin{aligned}
\int_0^\infty \frac{\log(1 + bu^2)}{\sinh^2 2\pi u}\, du &= \frac{1}{4}\int_0^\infty \frac{\log(1 + bu^2)}{\cosh^2 \pi u}\, \frac{du}{\sinh^2 \pi u} \\
&= \frac{1}{4}\int_0^\infty \frac{\log(1 + bu^2)}{\cosh^2 \pi u}\left(\frac{\cosh^2 \pi u}{\sinh^2 \pi u} - 1\right) du.
\end{aligned}$$

It follows that

$$\int_0^\infty \frac{\log(1 + bu^2)}{\cosh^2 \pi u}\, du = h(b, \pi) - 4h(b, 2\pi). \tag{2.26}$$

Now take $b = (x - \frac{1}{2})^{-2}$ to produce

$$\begin{aligned}
\mathbb{E}\log\left(1 + bL_B^2\right) &= \frac{\pi}{2}\int_0^\infty \frac{\log\left(1 + bu^2\right)}{\cosh^2 \pi u}\, du \\
&= \frac{\pi}{2}\left(h(b, \pi) - 4h(b, 2\pi)\right) \\
&= \log \frac{1}{\sqrt{b}} - 2\log \frac{2}{\sqrt{b}} - \psi\left(\frac{1}{\sqrt{b}}\right) + 2\psi\left(\frac{2}{\sqrt{b}}\right).
\end{aligned} \tag{2.27}$$

The duplication formula

$$\psi(2z) = \tfrac{1}{2}\psi(z) + \tfrac{1}{2}\psi\left(z + \tfrac{1}{2}\right) + \log 2, \tag{2.28}$$

that appears as entry 8.365.6 in [7], reduces (2.27) to the stated form. $\square$

3. The periodicity of the modified Bernoulli numbers B_{2n+1}^*

This section uses the umbral method to express the generating function of the modified Bernoulli numbers B_n^* in terms of the digamma function $\psi(x)$. The periodicity of B_{2n+1}^* in Theorem 1.1 follows from this computation. Zagier [21] establishes this result by using the expression

$$2\sum_{n=1}^{\infty} B_n^* x^n = \sum_{r=1}^{\infty} \frac{B_r}{r} \frac{x^r}{(1-x)^{2r}} - 2\log(1-x). \tag{3.1}$$

In the proofs given here the generating function employed admits an explicit expression.

Theorem 3.1. *The generating function of the sequence $\{B_n^*\}$ is given by*

$$F_{B^*}(z) := \sum_{n=1}^{\infty} B_n^* z^n = -\frac{1}{2}\log z - \frac{1}{2}\psi\left(z + 1/z - 1\right). \tag{3.2}$$

Proof. Start with

$$\begin{aligned} F_{B^*}(z) &= \sum_{n=1}^{\infty} z^n \sum_{r=0}^{\infty} \binom{n+r}{2r} \frac{B_r}{n+r} \\ &= \sum_{n=1}^{\infty} z^n \sum_{r=1}^{\infty} \binom{n+r}{2r} \frac{B_r}{n+r} + \sum_{n=1}^{\infty} z^n \binom{n}{0} \frac{B_0}{n}. \end{aligned}$$

The second term is $-\log(1-z)$. Interchanging the order of summation in the first term gives

$$\sum_{n=1}^{\infty} z^n \sum_{r=1}^{\infty} \binom{n+r}{2r} \frac{B_r}{n+r} = \sum_{r=1}^{\infty} \frac{B_r}{(2r)!} \sum_{n=r}^{\infty} z^n \frac{(n+r-1)!}{(n-r)!}$$

and the inner sum is identified as

$$\sum_{n=r}^{\infty} z^n \frac{(n+r-1)!}{(n-r)!} = \sum_{m=0}^{\infty} z^{m+r} \frac{(2r+m-1)!)}{m!} = (2r-1)! \frac{z^r}{(1-z)^{2r}}.$$

Therefore

$$F_{B^*}(z) = -\log(1-z) + \sum_{r=1}^{\infty} \frac{B_r}{2r} \frac{z^r}{(1-z)^{2r}}. \tag{3.3}$$

The rules of umbral calculus now give an expression for $F_{B^*}(z)$. The identity

$$\sum_{r=1}^{\infty} \frac{B_r}{2r} \frac{z^r}{(1-z)^{2r}} = -\text{eval}\left\{ \frac{1}{2} \log\left(1 - \frac{z\mathfrak{B}}{(1-z)^2} \right) \right\},$$

gives

$$F_{B^*}(z) = -\text{eval}\left\{ \frac{1}{2}\log\left((1-z)^2 - z\mathfrak{B} \right) \right\}. \tag{3.4}$$

Further reduction yields

$$\begin{aligned} \log\left((1-z)^2 - \mathfrak{B}z\right) &= \log z + \log\left(\frac{(1-z)^2}{z} - \mathfrak{B}\right) \\ &= \log z + \log\left(\frac{(1-z)^2}{z} + \mathfrak{B} + 1\right) \\ &= \log z + \log\left[\mathfrak{B}\left(1 + \frac{(1-z)^2}{z}\right)\right], \end{aligned}$$

using (2.9). The result now follows from Theorem 2.5. □

The generating function of B^*_{2n+1} is now obtained from Theorem 3.1. The proof presented here is similar to the one given in [6].

Theorem 3.2. *The generating function of the sequence of odd-order modified Bernoulli numbers is given by*

$$G_{B^*}(z) = \sum_{n=0}^{\infty} B^*_{2n+1} z^{2n+1} = \frac{3z^{11} - z^9 - z^7 + z^5 + z^3 - 3z}{4(z^{12}-1)}. \tag{3.5}$$

Proof. Start with

$$\frac{F_{B^*}(z) - F_{B^*}(-z)}{2} = \sum_{n=0}^{\infty} B^*_{2n+1} z^{2n+1}. \tag{3.6}$$

To evaluate $F_{B^*}(-z)$ use the relation (3.4) and the operational rule from Lemma 2.1 to obtain

$$\begin{aligned} 2F_{B^*}(-z) &= -\text{eval}\left\{\log\left((1+z)^2 + z\mathfrak{B}\right)\right\} \\ &= -\log z - \text{eval}\left\{\log\left(\frac{(1+z)^2}{z} + \mathfrak{B}\right)\right\} \\ &= -\log z - \text{eval}\left\{\log\left[\mathfrak{B}\left(\frac{(1+z)^2}{z}\right)\right]\right\} \\ &= -\log z - \psi\left(\frac{(1+z)^2}{z}\right). \end{aligned}$$

Therefore

$$\begin{aligned} G_{B^*}(z) &= \frac{F_{B^*}(z) - F_{B^*}(-z)}{2} \\ &= -\frac{1}{4}\left(\psi\left(1 + \frac{(1-z)^2}{z}\right) - \psi\left(\frac{(1+z)^2}{z}\right)\right) \\ &= \frac{1}{4}\psi\left(z + 1/z + 2\right) - \frac{1}{4}\psi\left(z + 1/z - 1\right). \end{aligned}$$

Now use the relation (entry 8.365 in [7])

$$\psi(z+m) = \psi(z) + \sum_{k=0}^{m-1} \frac{1}{z+k} \tag{3.7}$$

to obtain the result. □

Theorem 1.1 is now obtained as a consequence of Theorem 3.2.

Corollary 3.3. *The sequence of odd-order modified Bernoulli numbers* B_{2n+1}^{*} *is periodic of period* 6.

4. An inversion formula for the modified Bernoulli numbers

This section discusses an expression for the classical Bernoulli numbers B_n in terms of the modified ones B_n^*. The result appears already in [21], but the proof presented here extends directly to the polynomial case as stated in Theorem 8.1. The details are simplified by introducing a minor adjustment of B_n^*.

Lemma 4.1. *Define* $\overline{B}_n = B_n^* - 1/n$. *Then*

$$\overline{B}_n = \sum_{k=1}^{n} \binom{n+k-1}{n-k} \frac{B_k}{2k}. \tag{4.1}$$

Proof. The definition of B_n^* in (1.4) produces

$$\overline{B}_n = \sum_{k=1}^{n} \binom{n+k}{2k} \frac{B_k}{n+k}.$$

Then use

$$\binom{n+k}{2k} \frac{1}{n+k} = \frac{(n+k-1)!}{(2k-1)!(n-k)!} \frac{1}{2k}$$

to deduce the claim. □

The inversion result is stated next.

Theorem 4.2. *The sequence of Bernoulli numbers* B_n *are given in terms of the modified Bernoulli numbers* B_n^* *by*

$$B_n = 2n \sum_{k=1}^{n} (-1)^{n+k} \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right] B_k^* + 2(-1)^n \binom{2n-1}{n},$$

for $n \geq 1$.

Proof. The inversion formulas

$$a_n = \sum_{k=0}^{n} \binom{n+p+k}{n-k} b_k, \text{ and } b_n = \sum_{k=0}^{n} (-1)^{k+n} \left[\binom{2n+p}{n-k} - \binom{2n+p}{n-k-1} \right] a_k,$$

are given in [15, (23), p. 67]. Applying it to the sequence $\overline{B}_n$ gives

$$\frac{B_n}{2n} = \sum_{k=1}^{n} (-1)^{n+k} \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right] \overline{B}_k. \tag{4.2}$$

The result now follows from

$$\sum_{k=1}^{n} (-1)^{n+k} \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right] \frac{1}{k} = \frac{(-1)^n}{n} \binom{2n-1}{n}. \tag{4.3}$$

To prove the identity (4.3) write the summand as

$$\frac{1}{k} \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right] = \frac{1}{n} \binom{2n}{n-k} \tag{4.4}$$

and convert (4.3) to

$$\sum_{k=1}^{n}(-1)^k\binom{2n}{n-k} = -\binom{2n-1}{n}. \tag{4.5}$$

This follows directly from the basic sum

$$\sum_{k=0}^{2n}(-1)^j\binom{2n}{j} = 0. \tag{4.6}$$

□

5. A generating function for Zagier polynomials

This section gives the generating function of the Zagier polynomials

$$B_n^*(x) = \sum_{r=0}^{n}\binom{n+r}{2r}\frac{B_r(x)}{n+r}. \tag{5.1}$$

The proof is similar to Theorem 3.1, so just an outline of the proof is presented.

Theorem 5.1. *The generating function of the sequence $\{B_n^*(x)\}$ is given by*

$$F_{B^*}(x;z) = \sum_{n=1}^{\infty} B_n^*(x)z^n = -\frac{1}{2}\log z - \frac{1}{2}\psi\left(z+1/z-1-x\right). \tag{5.2}$$

Proof. The starting point is the polynomial variation of (3.4) in the form

$$\begin{aligned} F_{B^*}(x;z) &= -\text{eval}\left\{\frac{1}{2}\log\left((1-z)^2 - z\mathfrak{B}(x)\right)\right\} \\ &= -\text{eval}\left\{\frac{1}{2}\log\left((1-z)^2 - zx - z\mathfrak{B}\right)\right\} \\ &= -\text{eval}\left\{\frac{1}{2}\log\left(1-2z+z^2 - zx - z\mathfrak{B}\right)\right\} \\ &= -\frac{1}{2}\log z - \text{eval}\left\{\frac{1}{2}\log\left(1/z-2+z-x-\mathfrak{B}\right)\right\}. \end{aligned}$$

Now use $-\mathfrak{B} = \mathfrak{B}+1$ to obtain

$$F_{B^*}(x;z) = -\frac{1}{2}\log z - \frac{1}{2}\text{eval}\left\{\log\left(1/z+z-1-x+\mathfrak{B}\right)\right\}. \tag{5.3}$$

The final claim now follows from Theorem 2.5. □

Corollary 5.2. *The generating function of the sequence $\{(-1)^n B_n^*(x)\}$ is given by*

$$F_{B^*}(x;-z) := \sum_{n=1}^{\infty}(-1)^n B_n^*(x)z^n = -\frac{1}{2}\log z - \frac{1}{2}\psi\left(z+1/z+2+x\right). \tag{5.4}$$

Proof. Replacing z by $-z$ in the third line of the proof of Theorem 5.1 gives

$$\begin{aligned} F_{B^*}(x;-z) &= -\text{eval}\left\{\frac{1}{2}\log\left(1+2z+z^2+zx+z\mathfrak{B}\right)\right\} \\ &= -\frac{1}{2}\log z - \text{eval}\left\{\frac{1}{2}\log\left(z+1/z+2+x+\mathfrak{B}\right)\right\}. \end{aligned}$$

As before, the result now comes from Theorem 2.5. □

The next step is to provide analytic expressions for the generating functions of the subsequences $\{B_{2n+1}^*(x)\}$ and $\{B_{2n}^*(x)\}$. These formulas will be used in Section 12 to obtain information about these subsequences and in particular to discuss periodic subsequences in Theorem 1.2.

Corollary 5.3. *The generating functions of the odd and even parts of the sequence of Zagier polynomials are given by*

$$\sum_{n=0}^{\infty} B_{2n+1}^*(x) z^{2n+1} = \frac{1}{4}\psi\left(z + 1/z + 2 + x\right) - \frac{1}{4}\psi\left(z + 1/z - 1 - x\right), \tag{5.5}$$

and

$$\sum_{n=1}^{\infty} B_{2n}^*(x) z^{2n} = -\frac{1}{2}\log z - \frac{1}{4}\psi\left(z + 1/z + 2 + x\right) - \frac{1}{4}\psi\left(z + 1/z - 1 - x\right). \tag{5.6}$$

The results of Corollary 5.3 correspond to the analogue of the ordinary generating function for the Bernoulli polynomials. This is expressed in terms on the Hurwitz zeta function as stated in Theorem 5.4. The latter is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{5.7}$$

and it has the integral representation (see [18], page 76)

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-at}t^{s-1}}{1-e^{-t}}\, dt, \quad \operatorname{Re} s > 1, \operatorname{Re} a > 0. \tag{5.8}$$

The exponential generating function (1.6), because it is given by an elementary function, is employed more frequently.

Theorem 5.4. *The generating function of the Bernoulli polynomials $B_n(x)$ is given by*

$$\sum_{n=0}^{\infty} B_n(x) z^n = \frac{1}{z}\zeta\left(2, 1/z - x + 1\right). \tag{5.9}$$

Proof. The integral representation of the gamma function

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u}\, du, \quad \operatorname{Re} s > 0 \tag{5.10}$$

and the special value $\Gamma(n+1) = n!$ give

$$\sum_{n=0}^{\infty} B_n(x) z^n = \int_0^{\infty} e^{-u} \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} (zu)^n\, du. \tag{5.11}$$

The generating function (1.6) is used to produce

$$\sum_{n=0}^{\infty} B_n(x) z^n = z \int_0^{\infty} \frac{u}{1-e^{-zu}} e^{-(1-xz+z)u}\, du. \tag{5.12}$$

The change of variables $v = zu$ and (5.8) complete the proof. $\square$

6. Some arithmetic questions

There is marked difference in the arithmetical behavior of the numbers $B_n^*(j)$ according to the parity of n. For instance

$$\{B_n^*(0) : 1 \leq n \leq 10\} = \left\{ \frac{3}{4}, \frac{1}{24}, -\frac{1}{4}, -\frac{27}{80}, -\frac{1}{4}, -\frac{29}{1260}, \frac{1}{4}, \frac{451}{1120} \frac{1}{4}, -\frac{65}{264} \right\}$$

and

$$\{B_n^*(1) : 1 \leq n \leq 10\} = \left\{ \frac{5}{4}, \frac{25}{24}, \frac{5}{4}, \frac{133}{80}, \frac{9}{4}, \frac{3751}{1260}, \frac{15}{4}, \frac{4931}{1120} \frac{19}{4}, \frac{1255}{264} \right\}.$$

On the other hand, keeping n fixed and varying j gives

$$\{B_1^*(j) : 1 \leq j \leq 10\} = \left\{ \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}, \frac{17}{4}, \frac{19}{4} \frac{21}{4}, \frac{23}{4} \right\}$$

and

$$\{B_2^*(j) : 1 \leq j \leq 10\} = \left\{ \frac{25}{24}, \frac{61}{24}, \frac{109}{24}, \frac{169}{24}, \frac{241}{24}, \frac{325}{24}, \frac{421}{24}, \frac{529}{24} \frac{649}{24}, \frac{781}{24} \right\}.$$

This suggests that every element in the list $\{B_n^*(j) : j \geq 1\}$ has a denominator that is *independent* of j, therefore this value is also the denominator of the modified Bernoulli number B_n^*. Assume that this is true and define $\alpha(n)$ be this common value; that is,

$$\alpha(n) = \text{denominator}(B_n^*). \tag{6.1}$$

As usual, the parity of n plays a role in the results.

The next theorem shows, for the case n odd, that the function $\alpha(n)$ is well defined.

Theorem 6.1. *For $j \in \mathbb{Z}$, the values $4B_{2n+1}^*(j)$ are integers. That is,*

$$\alpha(2n+1) = 4. \tag{6.2}$$

Proof. The generating function (5.3) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 4B_{2n+1}^*(j) z^{2n+1} &= \psi\left(z + 1/z + j + 2\right) - \psi\left(z + 1/z - j - 1\right) \\
&= \psi\left(z + 1/z\right) + \sum_{k=0}^{j+1} \frac{1}{z + 1/z + k} - \psi\left(z + 1/z\right) + \\
&\qquad + \sum_{k=0}^{j} \frac{1}{z + 1/z - j - 1 + k} \\
&= \sum_{k=0}^{j+1} \frac{1}{z + 1/z + k} + \sum_{k=0}^{j+1} \frac{1}{z + 1/z - j - 1 + k} - \frac{1}{z + 1/z}.
\end{aligned}$$

Replace k by $j+1-k$ in the second sum to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} 4B_{2n+1}^{*}(j) z^{2n+1} &= \sum_{k=0}^{j+1}\left(\frac{1}{z+1/z+k}+\frac{1}{z+1/z-k}\right)-\frac{1}{z+1/z} \\ &= 2z\sum_{k=0}^{j+1}\frac{z^2+1}{(z^2+1)^2-k^2z^2}-\frac{z}{z^2+1} \\ &= 2z\sum_{k=1}^{j+1}\frac{z^2+1}{(z^2+1)^2-k^2z^2}+\frac{z}{z^2+1}.\end{aligned}$$

This implies $4B_{2n+1}^{*}(j) \in \mathbb{Z}$. ☐

Note 6.2. Arithmetic questions for the numbers $B_{2n}^{*}(j)$ seem to be more delicate. The values $\alpha(2n)$ seem to be divisible by 4 and the list of $\frac{1}{4}\alpha(2n)$ begins with

$$\{6,\ 20,\ 315,\ 280,\ 66,\ 3003,\ 78,\ 9520,\ 305235,\ 20900,\ 138,\ 19734,\ 6,\ 7540,\ 15575175\},$$

for $1 \leq n \leq 15$. The exact power of a prime p that divides $\alpha(2n)$ exhibits some interesting patterns. For instance, Figure 1 shows this function for $p=2$.

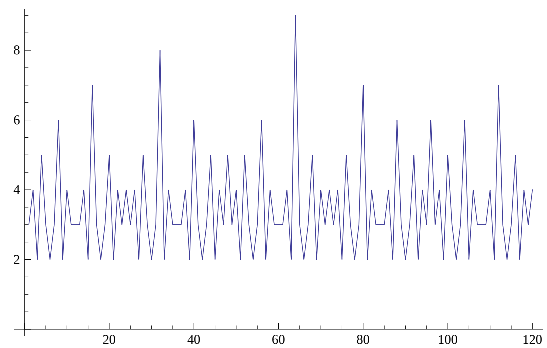

FIGURE 1. Power of 2 that divides denominator of $B_{2n}^{*}(j)$

The data suggests that the prime factors of $\alpha(2n)$ are bounded by $2n+1$. These questions will be addressed in a future paper.

7. Some auxiliary binomial sums

This section contains the proofs of two identities for some sums involving binomial coefficients. These sums will be used in the Section 8 to give an expression for the derivatives of Zagier polynomials. The identities given here are established using the method of creative telescoping described in [13].

Lemma 7.1. *For $n \in \mathbb{N}$,*

$$\sum_{r=1}^{n-1}(-1)^r \frac{2(r+1)}{n+r+1}\binom{2r-1}{r}\binom{n+r+1}{2r+2} = -\left\lfloor \frac{n}{2}\right\rfloor. \tag{7.1}$$

Proof. The summand on the left-hand side is written as

$$F(n,r)=\frac{(-1)^r\,(n+r)!}{2(2r+1)r!^2(n-r-1)!}.$$

Observe that $F(n,r)$ vanishes for $r<0$ or $r>n-1$. The method of Wilf-Zeilberger lends the companion function

$$G(n,r)=\frac{(-1)^{r+1}(n+r)!}{(n+1)(r-1)!^2(n-r+1)!} \tag{7.2}$$

together with the second order recurrence

$$F(n+2,r)-F(n,r)=G(n,r+1)-G(n,r). \tag{7.3}$$

Sum both sides over all integers r and check that the right-hand sum vanishes to produce

$$\sum_{r\in\mathbb{Z}}F(n+2,r)=\sum_{r\in\mathbb{Z}}F(n,r). \tag{7.4}$$

Define

$$f(n)=\sum_{r=0}^{n-1}F(n,r). \tag{7.5}$$

Then (7.4) gives $f(n+2)=f(n)$. The initial conditions $f(1)=1/2$ and $f(2)=0$ show that

$$f(n)=\begin{cases}\frac{1}{2} & \text{for } n \text{ odd}\\ 0 & \text{for } n \text{ even.}\end{cases} \tag{7.6}$$

The desired sum starts at $r=1$, so its value is $f(n)-F(n,0)$. Thus, $F(n,0)=n/2$ gives the result. ∎

Lemma 7.2. *For $n\in\mathbb{N}$ and $1\le k\le n-1$, define*

$$u(n,k)=\sum_{r=k}^{n-1}\frac{2(-1)^r r(r+1)}{n+r+1}\binom{n+r+1}{2r+2}\left[\binom{2r-1}{r-k}-\binom{2r-1}{r-k-1}\right]. \tag{7.7}$$

Then for n even

$$u(n,k)=\begin{cases}-k & \text{for } k \text{ odd,}\\ 0 & \text{for } k \text{ even}\end{cases} \tag{7.8}$$

and for n odd

$$u(n,k)=\begin{cases}0 & \text{for } k \text{ odd,}\\ k & \text{for } k \text{ even.}\end{cases} \tag{7.9}$$

Proof. A routine binomial simplification gives

$$u(n,k)=k\sum_{r=k}^{n-1}(-1)^r\binom{n+r}{2r+1}\binom{2r}{r-k}. \tag{7.10}$$

This motivates the definition $\bar{u}(n,k) = u(n,k)/k$ and the assertion in (7.8) and (7.9) amounts to showing

$$\bar{u}(n,k) = \begin{cases} +1 & \text{for } n \text{ odd}, k \text{ even} \\ -1 & \text{for } n \text{ even}, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{7.11}$$

The proof is similar to the one presented for Lemma 7.1. Introduce the functions $F(n,r,k) = (-1)^r \binom{n+r}{2r+1}\binom{2r}{r-k}$ and use the WZ-method to find the function

$$G(n,r,k) = \frac{2(n+1)(2r-1)(-1)^{r+1}}{(n+k+1)(n-k+1)} \binom{n+r}{2r-1}\binom{2r-2}{r-k-1} \tag{7.12}$$

companion to F and the equation

$$F(n+2,r,k) - F(n,r,k) = G(n,r+1,k) - G(n,r,k). \tag{7.13}$$

The argument is completed as before. $\square$

8. The derivatives of Zagier polynomials

Differentiation of the generating function for Bernoulli polynomials (1.6) gives the relation

$$\frac{d}{dx} B_n(x) = n B_{n-1}(x). \tag{8.1}$$

This section presents the analogous result for the Zagier polynomials. The proof employs an expression for $B_n(x)$ in terms of $B_n^*(x)$; that is the inversion of (1.5). The proof is identical to that of Theorem 4.2, so it is omitted.

Theorem 8.1. *The sequence of Bernoulli polynomial $B_n(x)$ is given in terms of the Zagier polynomials $B_n^*(x)$ by*

$$B_n(x) = 2n \sum_{k=1}^{n} (-1)^{n+k} \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right] B_k^*(x) + 2(-1)^n \binom{2n-1}{n},$$

for $n \geq 1$.

The analogue of (8.1) is established next.

Theorem 8.2. *The derivatives of the Zagier polynomials satisfy the relation*

$$\frac{d}{dx} B_n^*(x) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (2j-1) B_{2j-1}^*(x) \quad \text{for } n \text{ even}$$

and

$$\frac{d}{dx} B_n^*(x) = \frac{1}{2} + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} 2j B_{2j}^*(x) \quad \text{for } n \text{ odd.}$$

Proof. Differentiating (5.1) and using (8.1) gives

$$\begin{aligned} \frac{d}{dx} B_n^*(x) &= \sum_{r=0}^{n-1} \binom{n+r+1}{2r+2} \frac{r+1}{n+r+1} B_r(x) \\ &= \frac{n}{2} + \sum_{r=1}^{n-1} \binom{n+r+1}{2r+2} \frac{r+1}{n+r+1} B_r(x) \end{aligned}$$

for $n \geq 1$. The sum above (without the term $n/2$) is transformed using Theorem 8.1 to produce

$$\begin{aligned}(8.2)\quad & 2\sum_{r=0}^{n-1}(-1)^r\binom{2r-1}{r}\binom{n+r+1}{2r+2}\frac{r+1}{n+r+1}+\\ & 2\sum_{r=1}^{n-1}\frac{r(r+1)}{n+r+1}\binom{n+r+1}{2r+2}\sum_{k=1}^{r}(-1)^{k+r}\left[\binom{2r-1}{r-k}-\binom{2r-1}{r-k-1}\right]B_k^*(x).\end{aligned}$$

Denote the first sum by S_1 and the second one by S_2.

Lemma 7.1 shows that $S_1 = -\left\lfloor \frac{n}{2} \right\rfloor$. To evaluate S_2, reverse the order of summation to obtain

$$S_2 = \sum_{k=1}^{n-1}(-1)^k u(n,k) B_k^*(x), \tag{8.3}$$

with $u(n,k)$ defined in (7.7). This gives

$$\frac{d}{dx}B_n^*(x) = \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + \sum_{k=1}^{n-1}(-1)^k u(n,k) B_k^*(x), \tag{8.4}$$

and the proof now follows from the values of $u(n,k)$ given in Lemma 7.2. □

9. Some basics on Chebyshev polynomials

This section contains some basic information about the Chebyshev polynomials of first and second kind, denoted by $T_n(x)$ and $U_n(x)$, respectively. These properties will be used to establish some results on Zagier polynomials and the relation between these two families of polynomials will be clarified in Section 10.

The Chebsyhev polynomials of the first kind T_n are defined by

$$T_n(\cos\theta) = \cos n\theta, \quad n \in \mathbb{N} \cup \{0\} \tag{9.1}$$

and the companion Chebyshev polynomial of the second kind U_n by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{9.2}$$

Their generating functions are given by

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2} \tag{9.3}$$

and

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2xt+t^2}. \tag{9.4}$$

The even part of this series, given by

$$\sum_{n=0}^{\infty} U_{2n}(x)t^{2n} = \frac{1+t^2}{1+2(1-2x^2)t^2+t^4}, \tag{9.5}$$

will be used in Section 11. Many properties of these families may be found in [12]. The formulas for the generating functions also appear in [16, 22 : 3 : 8, page 199].

Differentiation of the generating function for $T_n(x)$ gives the basic identity

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \tag{9.6}$$

The expression

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}, \tag{9.7}$$

will be used in the arguments presented below.

The Chebyshev polynomials have a hypergeometric representation in the form

$$\begin{aligned} T_n(x) &= {}_2F_1\left(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}\right) \\ U_n(x) &= (n+1)\,{}_2F_1\left(-n, n+2; \tfrac{3}{2}; \tfrac{1-x}{2}\right). \end{aligned} \tag{9.8}$$

These appear in [12, p. 394, equation (15.9.5) and(15.9.6)].

The next statement is an expression of the Chebyshev polynomial T_n that will be used to establish, in Theorem 11.1, a symmetry property of the Zagier polynomials.

Lemma 9.1. *The Chebyshev polynomial $T_n(x)$ satisfies*

$$\sum_{r=0}^{n} \binom{n+r}{2r} \frac{x^r}{n+r} = \frac{1}{n} T_n\left(\frac{x}{2}+1\right). \tag{9.9}$$

Proof. The representation (9.8) and $\left(\frac{1}{2}\right)_r = 2^{-2r}(2r)!/r!$ yield

$$\begin{aligned} T_n\left(\frac{x}{2}+1\right) &= {}_2F_1\left(-n,\, n;\, \tfrac{1}{2},\, -\tfrac{x}{4}\right) \\ &= \sum_{r=0}^{n} \frac{(-n)_r (n)_r}{(\frac{1}{2})_r} \frac{(-x/4)^r}{r!} \\ &= \sum_{r=0}^{n} \{n(n-1)\cdots(n-(r-1))\}\{n(n+1)\cdots(n+r-1)\} \frac{x^r}{(2r)!} \\ &= n\sum_{r=0}^{n} \frac{(n-r)!\,\{(n-r+1)(n-r+2)\cdots(n+r-1)\}}{(2r)!\,(n-r)!} x^r \\ &= n\sum_{r=0}^{n} \frac{(n+r-1)!}{(2r)!\,(n-r)!} x^r. \end{aligned}$$

This verifies the claim. □

Lemma 9.2. *The Zagier polynomial $B_n^*(x)$ is related to the Chebyshev polynomial $T_n(x)$ via*

$$B_n^*(x) = \text{eval}\left\{\frac{1}{n}T_n\left(\frac{\mathfrak{B}(x)}{2}+1\right)\right\} = \text{eval}\left\{\frac{1}{n}T_n\left(\frac{\mathfrak{B}+x+2}{2}\right)\right\}.$$

Proof. This is simply the umbral version of (9.9). □

Lemma 9.3. *The Zagier polynomial $B_n^*(x)$ is given by*

$$B_n^*(x) = \frac{1}{n}\mathbb{E}\left[T_n\left(\frac{x}{2}+\frac{1}{2}iL_B+\frac{3}{4}\right)\right]. \tag{9.10}$$

Proof. The result now follows from Lemma 9.2, Theorem 2.3 and the umbral rule (2.9). □

The result of Lemma 9.3 is now used to give an umbral proof of Theorem 8.2.

Proof. The computation is simpler with $\tilde{B}_n(x) = B_n^*\left(x - \frac{3}{2}\right)$. Differentiate the statement of Lemma 9.3 to obtain

$$\frac{d}{dx}\tilde{B}_n(x) = \frac{1}{2n}\mathbb{E}\left[T_n'\left(\frac{x+iL_B}{2}\right)\right] = \frac{1}{2}\mathbb{E}\left[U_{n-1}\left(\frac{x+iL_B}{2}\right)\right]. \tag{9.11}$$

In the case of even degree, this gives

$$\frac{d}{dx}\tilde{B}_{2n}(x) = \frac{1}{2}\mathbb{E}\left[U_{2n-1}\left(\frac{x+iL_B}{2}\right)\right], \tag{9.12}$$

and using the identity

$$U_{2n-1}(x) = 2\sum_{k=1}^{n} T_{2k-1}(x) \tag{9.13}$$

that is entry 18.18.33 in [12], it follows that

$$\frac{d}{dx}\tilde{B}_{2n}(x) = \mathbb{E}\left[\sum_{k=1}^{n} T_{2k-1}\left(\frac{x+iL_B}{2}\right)\right] = \sum_{k=1}^{n}(2k-1)\tilde{B}_{2k-1}(x), \tag{9.14}$$

as claimed. The same argument works for n odd using [12, 18.18.32]:

$$U_{2n}(x) = 2\sum_{k=0}^{n} T_{2k}(x) - 1. \tag{9.15}$$

□

10. The Zagier-Chebyshev connection

In [21], after the proof of the identity

$$2B_{2n}^* = \binom{-3}{n} + \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{B_{2r}}{n+r}, \tag{10.1}$$

the author remarks that the second term has a *pleasing similarity* to the equation (1.4). This section contains a representation of the Zagier polynomials $B_n^*(x)$ in terms of the Chebyshev polynomials of the second kind $U_n(x)$. The expressions contain terms that also have pleasing similarity to the definition of Zagier polynomials. The results are naturally divided according to the parity of n.

Theorem 10.1. *The Zagier polynomials are given by*

$$2B_{2n}^*(x) = \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{B_{2r}(x)}{n+r} + U_{2n-1}\left(\frac{x}{2}\right) + U_{2n-1}\left(\frac{x+1}{2}\right) \tag{10.2}$$

and

$$2B_{2n+1}^*(x) = \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r+1}{2r+1}\frac{B_{2r+1}(x)}{n+r+1} + U_{2n}\left(\frac{x}{2}\right) + U_{2n}\left(\frac{x+1}{2}\right). \tag{10.3}$$

Proof. The proof is presented for the even degree case, the proof is similar for odd degree.

Theorem 5.1 gives the generating function for $B_n^*(x)$. Its even part yields

$$2\sum_{n=1}^{\infty} B_{2n}^*(x)z^{2n} = -\frac{1}{2}\log z - \frac{1}{2}\psi(1/z+z-x-1) - \frac{1}{2}\log(-z) - \frac{1}{2}\psi(-1/z-z-x-1).$$

The functional equation $\psi(t+1) = \psi(t) + 1/t$ gives

$$2\sum_{n=1}^{\infty} B_{2n}^{*}(x)z^{2n} = H(x,z) + H(x,-z) + \frac{1}{2}\left(\frac{1}{1/z+z+x+3} + \frac{1}{1/z+z+x+2} - \frac{1}{1/z+z-x-3} - \frac{1}{1/z+z-x-2} \right) \tag{10.4}$$

with

$$H(x,z) = -\frac{1}{2}\left(\log z + \psi(1/z+z-x-3) \right). \tag{10.5}$$

The umbral method and Theorem 2.5 give

$$\begin{aligned}
2H(x,z) &= -\log z - \text{eval}\left(\log(1/z+z-x-3+\mathfrak{B}) \right) \\
&= -\log z - \text{eval}\left(\log(1/z+z-x-4-\mathfrak{B}) \right) \\
&= -\text{eval}\left(\log(1+z^2-4z-zx-x\mathfrak{B}) \right) \\
&= -\log(1+z^2) - \text{eval}\left(\log(1 - \frac{z\mathfrak{B}(x+4)}{1+z^2}) \right) \\
&= -\log(1+z^2) + \text{eval}\left(\sum_{r=1}^{\infty} \frac{(z\mathfrak{B}(x+4))^r}{r(1+z^2)^r} \right) \\
&= -\log(1+z^2) + \sum_{r=1}^{\infty} \frac{B_r(x+4)z^r}{r(1+z^2)^r}.
\end{aligned}$$

Therefore

$$H(x,z) + H(x,-z) = -\log(1+z^2) + \sum_{r=1}^{\infty} \frac{B_{2r}(x+4)\, z^{2r}}{2r(1+z^2)^{2r}}.$$

Now observe that

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{B_{2r}(x+4)\, z^{2r}}{2r(1+z^2)^{2r}} &= \sum_{r=1}^{\infty} \frac{z^{2r} B_{2r}(x+4)}{2r} \sum_{n=0}^{\infty} \frac{(2r)_n(-z^2)^n}{n!} \\
&= \sum_{r=1}^{\infty} \frac{(-1)^r B_{2r}(x+4)}{2r} \sum_{n=r}^{\infty} (-1)^n \binom{n+r-1}{2r-1} z^{2n} \\
&= \sum_{n=1}^{\infty} \left(\sum_{r=1}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}(x+4)}{n+r} \right) z^{2n}.
\end{aligned}$$

This produces

$$\sum_{r=1}^{\infty} \frac{B_{2r}(x+4)\, z^{2r}}{2r(1+z^2)^{2r}} - \log(1+z^2) = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}(x+4)}{n+r} \right) z^{2n}.$$

The equation (10.4) now gives

$$2\sum_{n=1}^{\infty} B_{2n}^{*}(x)z^{2n} = \sum_{n=1}^{\infty} \left(\sum_{r=0}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}(x+4)}{n+r} \right) z^{2n} + \frac{1}{2}\left(\frac{1}{1/z+z+x+3} + \frac{1}{1/z+z+x+2} - \frac{1}{1/z+z-x-3} - \frac{1}{1/z+z-x-2} \right). \tag{10.6}$$

The generating function for $U_{n-1}(x)$, given in (9.4), is written as

$$\sum_{n=1}^{\infty} U_{n-1}(x) z^n = \frac{1}{1/z + z - 2x} \tag{10.7}$$

and the rational function appearing in (10.6) is expressed as

$$\sum_{n=1}^{\infty} \left(U_{n-1}\left(\frac{-x-3}{2} \right) + U_{n-1}\left(\frac{-x-2}{2} \right) - U_{n-1}\left(\frac{x+3}{2} \right) - U_{n-1}\left(\frac{x+2}{2} \right) \right) z^n.$$

Using the fact that $U_n(x)$ has the same parity as n it simplifies to

$$2\sum_{n=1}^{\infty} \left(U_{2n-1}\left(\frac{-x-3}{2} \right) + U_{2n-1}\left(\frac{-x-2}{2} \right) \right) z^{2n}.$$

Therefore

$$2B_{2n}^{*}(x) = \sum_{r=0}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}(x+4)}{n+r} + U_{2n-1}\left(\frac{-x-3}{2} \right) + U_{2n-1}\left(\frac{-x-2}{2} \right).$$

Finally, replace x by $-x-3$, use Theorem 11.1 and the symmetry of Bernoulli polynomials $B_{2n}(1-x) = B_{2n}(x)$, to obtain the result. □

The next result gives a representation for the difference of two Zagier polynomials in terms of the Chebyshev polynomials $U_n(x)$.

Lemma 10.2. *The Zagier polynomials satisfy*

$$B_n^{*}(x+1) = B_n^{*}(x) + \frac{1}{2} U_{n-1}\left(\frac{x}{2} + 1 \right). \tag{10.8}$$

It can be extended to

$$B_n^{*}(x) - B_n^{*}(x-k) = \frac{1}{2} \sum_{j=1}^{k} U_{n-1}\left(\frac{x-j}{2} + 1 \right). \tag{10.9}$$

Proof. Apply Lemma 2.2 and the representation of $B_n^{*}(x)$ in Lemma 9.2. □

Note 10.3. The sum in (10.2) equals

$$\sum_{r=0}^{n} (-1)^{n+r} \binom{n+r}{2r} \frac{B_{2r}(x)}{n+r} = 2B_{2n}^{*}(x-2). \tag{10.10}$$

The proof of this fact is given below. First, use it to express (10.2) as

$$B_{2n}^{*}(x) - B_{2n}^{*}(x-2) = \frac{1}{2} \left(U_{2n-1}\left(\frac{x}{2} \right) + U_{2n-1}\left(\frac{x+1}{2} \right) \right). \tag{10.11}$$

In this form, it can be extended directly to

$$B_{2n}^{*}(x) - B_{2n}^{*}(x-2k) = \frac{1}{2} \left(\sum_{j=0}^{k-1} U_{2n-1}\left(\frac{x-2j}{2} \right) + U_{2n-1}\left(\frac{x-2j+1}{2} \right) \right). \tag{10.12}$$

The proof of (10.10) is given next. The identity

$$T_{2n}(x) = (-1)^n T_n(1-2x^2) \tag{10.13}$$

that appears in [4, 7.2.10(7), page 550] is used in the proof. Start with

$$\begin{aligned}
\sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{B_{2r}(x)}{n+r} &= \text{eval}\left\{(-1)^n\sum_{r=0}^{n}\binom{n+r}{2r}\frac{(-\mathfrak{B}^2(x))^r}{n+r}\right\} \\
&= \text{eval}\left\{\frac{(-1)^n}{n}T_n\left(-\frac{\mathfrak{B}^2(x)}{2}+1\right)\right\} \\
&= \text{eval}\left\{\frac{(-1)^n}{n}T_n\left(1-2\left(\frac{\mathfrak{B}(x)}{2}\right)^2\right)\right\} \\
&= 2\,\text{eval}\left\{\frac{1}{2n}T_{2n}\left(\frac{\mathfrak{B}(x)}{2}\right)\right\} \\
&= 2\,\text{eval}\left\{\frac{1}{2n}T_{2n}\left(\frac{\mathfrak{B}(x)-2}{2}+1\right)\right\} \\
&= 2\,\text{eval}\left\{\frac{1}{2n}T_{2n}\left(\frac{\mathfrak{B}(x-2)}{2}+1\right)\right\} \\
&= 2\sum_{r=0}^{2n}\binom{2n+r}{2r}\frac{B_r(x-2)}{2n+r} \\
&= 2B_{2n}^*(x-2).
\end{aligned}$$

A special case of Theorem 10.1 gives a simple proof of (10.1).

Corollary 10.4. *The modified Bernoulli numbers B_{2n}^* are given by*

$$2B_{2n}^* = \left(\frac{-3}{n}\right) + \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{B_{2r}}{n+r}. \tag{10.14}$$

Here $\left(\frac{\bullet}{n}\right)$ is the Jacobi symbol.

Proof. Let $x=0$ in Theorem 10.1, use the value $U_{2n-1}(0)=0$ and observe that

$$U_{2n-1}\left(\tfrac{1}{2}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3, \\ -1 & \text{if } n \equiv -1 \bmod 3, \\ 0 & \text{if } n \equiv 0 \bmod 3, \end{cases} \tag{10.15}$$

can be written as $U_{2n-1}\left(\frac{1}{2}\right) = \left(\frac{-3}{n}\right)$. $\square$

The next result appears in [21].

Corollary 10.5. *Let $n \in \mathbb{N}$. Then*

$$B_{2n}^* + n = \sum_{r=0}^{2n}\binom{2n+r}{2r}\frac{(-1)^r B_r}{2n+r}. \tag{10.16}$$

Proof. The right-hand side of (10.16) is $B_{2n}^*(1)$. Therefore, the statement becomes $B_{2n}^* + n = B_{2n}^*(1)$. This is established by letting $x=0$ in (10.8) and the value

$$U_{2n-1}(1) = \lim_{\theta\to 0}\frac{\sin 2n\theta}{\sin\theta} = 2n. \tag{10.17}$$

$\square$

Theorem 10.1 is now used to produce another proof of Theorem 1.1.

Corollary 10.6. *The modified Bernoulli numbers B^*_{2n+1} are given by*

$$B^*_{2n+1} = \frac{(-1)^n}{4} + \frac{1}{2}U_{2n}(\tfrac{1}{2}) = \frac{(-1)^n}{4} + \frac{1}{\sqrt{3}}\sin\left(\frac{(2n+1)\pi}{3}\right). \tag{10.18}$$

In particular, B^*_{2n+1} is periodic of period 6.

Proof. Put $x=0$ in (10.3) and observe that only the term $r=0$ survives in the sum. Now use the value $U_{2n}(0)=(-1)^n$ and let $\theta=\pi/3$ in (9.2) to get the result. □

Corollary 10.7. *For $n \in \mathbb{N}$*

$$2B^*_{2n+1}\left(\tfrac{1}{2}\right) = U_{2n}\left(\tfrac{1}{4}\right) + U_{2n}\left(\tfrac{3}{4}\right). \tag{10.19}$$

Proof. Let $x=\frac{1}{2}$ in (10.3) and use the fact that $B_j(\frac{1}{2}) = -(1-2^{1-j})B_j$. □

11. A reflection symmetry of the Zagier polynomials

The classical Bernoulli polynomials $B_n(x)$ exhibit symmetry with respect to the line $x=\frac{1}{2}$ in the form

$$B_n(1-x) = (-1)^nB_n(x). \tag{11.1}$$

This section describes the corresponding property for the Zagier polynomials: their symmetry is with respect to the line $x=-\frac{3}{2}$.

Theorem 11.1. *The Zagier polynomials satisfy the relation*

$$B^*_n(-x-3) = (-1)^nB^*_n(x). \tag{11.2}$$

Proof. The first proof uses the generating function $F_{B^*}(x,z)$. Replacing (x,z) by $(-x-3,-z)$ in the second line of the proof of Lemma 5.1 gives

$$\begin{aligned} F_{B^*}(-x-3,-z) &= -\text{eval}\left\{\frac{1}{2}\log\left((1+z)^2 - z(x+3) + z\mathfrak{B}\right)\right\} \\ &= -\frac{1}{2}\log z - \text{eval}\left\{\frac{1}{2}\log\left(z+1/z+\mathfrak{B}-1-x\right)\right\} \\ &= F_{B^*}(x,z). \end{aligned}$$

This proves the statement. □

A second proof of Theorem 11.1 uses the expression for the Zagier polynomials $B^*_n(x)$ in terms of the Chebyshev polynomials $T_n(x)$. Indeed, using $T_n(z) = (-1)^nT_n(z)$,

$$\begin{aligned} B^*_n(-x-3) &= \frac{1}{n}T_n\left(\frac{-x-3+\mathfrak{B}}{2}+1\right) \\ &= \frac{1}{n}T_n\left(\frac{-x+\mathfrak{B}}{2}-\frac{1}{2}\right) \\ &= \frac{(-1)^n}{n}T_n\left(\frac{x-\mathfrak{B}}{2}+\frac{1}{2}\right) \\ &= \frac{(-1)^n}{n}T_n\left(\frac{x+\mathfrak{B}+1}{2}+\frac{1}{2}\right) \\ &= \frac{(-1)^n}{n}T_n\left(\frac{x+\mathfrak{B}}{2}+1\right) \\ &= (-1)^nB^*_n(x). \end{aligned}$$

The rest of the section gives a third proof of Theorem 11.1.

Proof. The induction hypothesis states that $B_m^*(-x-3) = (-1)^m B_m^*(x)$ for all $m < n$. The discussion is divided according to the parity of n.

Case 1. For n is even, Theorem 8.2 gives

$$\begin{aligned} \frac{d}{dx}B_n^*(-x-3) &= -\sum_{j=1}^{n/2}(2j-1)B_{2j-1}^*(-x-3) \\ &= -\sum_{j=1}^{n/2}(2j-1)(-1)^{2j-1}B_{2j-1}^*(x) \\ &= \sum_{j=1}^{n/2}(2j-1)B_{2j-1}^*(x) \\ &= \frac{d}{dx}B_n^*(x). \end{aligned}$$

It follows that $B_n^*(-x-3)$ and $B_n^*(x)$ differ by a constant. Now evaluate at $x = -\frac{3}{2}$ to see that this constant vanishes.

Case 2. Now assume n is odd. The previous argument now gives

$$B_n^*(-x-3) = -B_n^*(x) + C_n \tag{11.3}$$

for some constant C_n. It remains to show $C_n = 0$.

Iterating the relation

$$B_n(x+1) = B_n(x) + nx^{n-1} \tag{11.4}$$

gives

$$B_n(x+3) = B_n(x) + nx^{n-1} + n(x+1)^{n-1} + n(x+2)^{n-1}. \tag{11.5}$$

Replace $x = -\frac{3}{2}$ in (11.3) and in (11.5) and observe that

$$C_n = 2\sum_{r=0} \frac{\binom{n+r}{2r}}{n+r}\left[B_r\left(\tfrac{1}{2}\right) - r\left(-\tfrac{3}{2}\right)^{r-1} - r\left(-\tfrac{1}{2}\right)^{r-1}\right]. \tag{11.6}$$

Thus, to show $C_n = 0$, it is required to prove

$$\sum_{r=0}^{n} \frac{\binom{n+r}{2r}}{n+r}B_r\left(\tfrac{1}{2}\right) = \sum_{r=0}^{n} \frac{\binom{n+r}{2r}}{n+r}\left[r\left(-\tfrac{3}{2}\right)^{r-1} + r\left(-\tfrac{1}{2}\right)^{r-1}\right]. \tag{11.7}$$

The left-hand side is nothing but $B_n^*\left(\frac{1}{2}\right)$. The right-hand side is $V_n'(-\frac{3}{2}) + V_n'(-\frac{1}{2})$, where

$$V_n(x) = \sum_{r=0}^{n} \frac{\binom{n+r}{2r}}{n+r}x^r. \tag{11.8}$$

Lemma 9.1 shows that

$$V_n(x) = \frac{1}{n}T_n\left(\frac{x}{2}+1\right). \tag{11.9}$$

Hence it suffices to show that

$$2B_n^*\left(\tfrac{1}{2}\right) = U_{n-1}\left(\tfrac{1}{4}\right) + U_{n-1}\left(\tfrac{3}{4}\right). \tag{11.10}$$

This is the result of Corollary 10.7. $\square$

Note 11.2. We note that unlike (10.2), the formula in (10.3), of which (11.10) is a special case, does not use the symmetry $B_{2n+1}^{*}(-x-3) = -B_{2n+1}^{*}(x)$ in its proof.

12. Values of Zagier polynomials that yield periodic sequences

The original observation of Zagier, that $B_{2n+1}^{*} = B_{2n+1}^{*}(0)$ is a periodic sequence (with period 6 and values $\{\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\}$) is extended here to other values of x. The first part of the discussion is to show that periodicity of $B_{2n+1}^{*}(x)$ implies that $2x$ is an integer.

The discussion begins with an elementary statement.

Lemma 12.1. *The sequence $\{a_n\}$ is periodic, with minimal period p, if and only if its generating function*

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \tag{12.1}$$

is a rational function of z such that, when written in reduced form, the denominator has the form $D(z) = 1 - z^p$.

Special values of $B_{2n+1}^{*}(x)$. The case considered here discusses values of x such that $\{B_{2n+1}^{*}(x)\}$ is a periodic sequence. The generating function of this sequence is given in (5.5) as

$$\sum_{n=0}^{\infty} B_{2n+1}^{*}(x) z^{2n+1} = \frac{1}{4}\psi\left(z + 1/z + 2 + x\right) - \frac{1}{4}\psi\left(z + 1/z - 1 - x\right). \tag{12.2}$$

Proposition 12.2. *Let $b \in \mathbb{R}$ be fixed. Then*

$$\psi(t+b) - \psi(t) = R(t) \tag{12.3}$$

for some rational function $R(t)$ if and only if $b \in \mathbb{Z}$.

Proof. Assume $b \in \mathbb{Z}$. It is clear that b may be assumed positive. Iteration of $\psi(t+1) = \psi(t) + 1/t$ yields

$$\psi(t+b) = \psi(t) + \sum_{k=0}^{b-1} \frac{1}{t+k}. \tag{12.4}$$

Therefore $\psi(t+b) - \psi(t)$ is a rational function. To prove the converse, assume (12.3) holds for some rational function R. Integrating both sides with respect to t gives

$$\ln\Gamma(t+b) - \ln\Gamma(t) = R_1(t) + \ln R_2(t) + C_1 \tag{12.5}$$

for a pair of rational functions R_1, R_2 (coming from the integration of $R(t)$) and a constant of integration C_1. It follows that

$$\frac{\Gamma(t+b)}{C_2 R_2(t)\Gamma(t)} = e^{R_1(t)}. \tag{12.6}$$

The singularities of the left-hand side are (at most) poles. On the other hand, the presence of a pole of $R_1(t)$ produces an essential singularity for the right-hand side of (12.6). It follows that $R_1(t)$ is a polynomial. Comparing the behavior of (12.6) as $t \to \pm\infty$ shows that R_1 must be a constant; that is,

$$\Gamma(t+b) = C_3 R_2(t)\Gamma(t). \tag{12.7}$$

The set equality

$$\{b-k:\, k\in\mathbb{N}\} = \{-k:\, k\in\mathbb{N}\}\cup\{t_1,\, t_2,\cdots,t_r\} \tag{12.8}$$

where t_i are the poles of R comes from comparing poles in (12.7). Now take $k\in\mathbb{N}$ sufficiently large so that $b-k\neq t_i$. Then $b-k=-k_1$ for some $k_1\in\mathbb{N}$. Therefore $b=k-k_1\in\mathbb{Z}$, as claimed. □

The next lemma deals with the transition from the variable z to $z+1/z$.

Lemma 12.3. *Assume $R(z)$ is a rational function that satisfies $R(z)=R(1/z)$. Then R is a function of $1/z+z$ only.*

Proof. It is assumed that

$$R(z) = \frac{a_0+a_1z+\cdots+a_nz^n}{b_0+b_1z+\cdots+b_mz^m} = \frac{a_0+a_1/z+\cdots+a_n/z^n}{b_0+b_1/z+\cdots+b_m/z^m}. \tag{12.9}$$

Now use the fact that

$$\frac{u}{v}=\frac{U}{V} \text{ implies } \frac{u}{v}=\frac{U}{V}=\frac{u+U}{v+V} \tag{12.10}$$

to conclude that

$$R(z) = \frac{2a_0+a_1(z+1/z)+a_2(z^2+1/z^2)+\cdots+a_n(z^n+1/z^n)}{2b_0+b_1(z+1/z)+b_2(z^2+1/z^2)+\cdots+b_m(z^m+1/z^m)}.$$

The conclusion follows from the fact that z^j+1/z^j is a polynomial in $z+1/z$. This is given in entry 1.331.3 of [7]. □

Theorem 12.4. *The generating function*

$$\sum_{n=0}^{\infty} B^*_{2n+1}(x)z^{2n+1} \tag{12.11}$$

is a rational function of z if and only if $2x\in\mathbb{Z}$.

Proof. Assume (12.11) is a rational function of z. Then (5.5) implies that

$$\psi(z+1/z+2+x)-\psi(z+1/z-1-x) = A(z) \tag{12.12}$$

with A a rational function of z. The left-hand side of (12.12) is invariant under $z\mapsto 1/z$, therefore Lemma 12.3 shows that $A(z)=B(z+1/z)$, for some rational function B. Now rewrite (12.12) as

$$\psi(t+2x+3)-\psi(t) = B(t+1+x) \tag{12.13}$$

with $t=z+1/z-1-x$. Proposition 12.2 shows that $2x\in\mathbb{Z}$.

To establish the converse, assume $2x\in\mathbb{Z}$. The identity (5.5) shows that

$$4\sum_{n=0}^{\infty} B^*_{2n+1}(x)z^{2n+1} = \psi(t+2x+3)-\psi(t) \tag{12.14}$$

with $t=z+1/z-1-x$. Proposition 12.2 shows that $\psi(t+2x+3)-\psi(t)$ is a rational function of t and hence a rational function of z. □

Corollary 12.5. *Assume the sequence $\{B^*_{2n+1}(x)\}$ is periodic. Then $2x\in\mathbb{Z}$.*

Proof. The hypothesis implies that the generating function in (12.11) is a rational function. Theorem 12.4 gives the conclusion. □

The quest for values of x that produce periodic sequences $B^*_{2n+1}(x)$ is now reduced to the set $\mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{2}\right)$. The symmetry given in Theorem 11.1 implies that one may assume $x \leq -\frac{3}{2}$.

12.1. **Integer values of x.** The nature of the sequence $\{B^*_{2n+1}(x)\}$ is discussed next for $x = k \in \mathbb{Z}$.

Theorem 12.6. *Let $n \in \mathbb{N}$ and $k \geq 3$. Then*

$$B^*_{2n+1}(-k) = -\frac{1}{4} U_{2n}(0) - \frac{1}{2} \sum_{j=1}^{k-2} U_{2n}\left(\frac{j}{2}\right). \tag{12.15}$$

Proof. This is just a special case of (10.9) with $x = 0$. Use (10.18) and the fact that $U_{2n}(0) = (-1)^n$. □

The next step is to show that $\{B^*_{2n+1}(-k)\}$ is not periodic for $k \geq 5$.

Lemma 12.7. *Assume $j \geq 3$. Then $U_{2n}\left(\frac{j}{2}\right) > 0$.*

Proof. This comes directly from (9.7). □

Proposition 12.8. *The sequence $\left\{B^*_{2n+1}(-k)\right\}$ is not periodic for $k \geq 5$.*

Proof. The identity (12.15) is written as

$$\begin{aligned} -2B^*_{2n+1}(-k) &= \frac{U_{2n}(0)}{2} + U_{2n}(1/2) + U_{2n}(1) + U_{2n}(3/2) + \sum_{j=4}^{k-2} U_{2n}\left(\frac{j}{2}\right) \\ &\geq \frac{U_{2n}(0)}{2} + U_{2n}(1/2) + U_{2n}(1) + U_{2n}(3/2). \end{aligned}$$

The value

$$U_{2n}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{2n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{2n+1}\right] \tag{12.16}$$

shows that $\left\{B^*_{2n+1}(-k)\right\}$ is not bounded. To obtain (12.16), use $x = \frac{3}{2}$ in (9.7). □

The next result shows that, after a linear modification, the case $k = -4$ produces another periodic example.

Proposition 12.9. *The sequence $\left\{B^*_{2n+1}(-4) + n\right\}$ is 6-periodic.*

Proof. The value $k = 4$ in (12.15) gives

$$B^*_{2n+1}(-4) = -\frac{1}{4} U_{2n}(0) - \frac{1}{2} U_{2n}\left(\tfrac{1}{2}\right) - \frac{1}{2} U_{2n}(1). \tag{12.17}$$

The values $U_{2n}(0) = (-1)^n$ is 2-periodic and

$$U_{2n}(\tfrac{1}{2}) = \frac{2}{\sqrt{3}} \sin\left((2n+1)\pi/3\right) \tag{12.18}$$

is 3-periodic (with values $0, -1, +1$). The expression (9.2), in the limit as $\theta \to 0$, gives $U_{2n}(1) = 2n + 1$. The proof is complete. □

Corollary 12.10. *The sequence $\left\{B^*_{2n+1}(-3)\right\}$ is 6-periodic.*

Proof. Choose $k = 3$ in (12.15) to obtain

$$B^*_{2n+1}(-3) = -\frac{1}{4}U_{2n}(0) - \frac{1}{2}U_{2n}\left(\frac{1}{2}\right). \tag{12.19}$$

As in Proposition 12.9, $U_{2n}(0)$ is of period 2 and $U_{2n}\left(\frac{1}{2}\right)$ is of period 3. □

Proposition 12.11. *The sequence* $\{B^*_{2n+1}(-2)\}$ *is* 2*-periodic:*

$$B^*_{2n+1}(-2) = \frac{(-1)^{n+1}}{4}. \tag{12.20}$$

Proof. Let $k = 2$ in Theorem 12.6. □

The rest of the integer values x are obtained by the symmetry rule given in Theorem 11.1. The study of the structure of the sequences $B^*_{2n+1}(k)$ has been completed. The details are summarized in the next statement.

Theorem 12.12. *Let* $k \in \mathbb{Z}$. *Then*
a) $\{B^*_{2n+1}(k)\}$ *is exponentially unbounded if* $k \geq 2$ *or* $k \leq -5$*;*
b) $\{B^*_{2n+1}(k) + n\}$ *is* 6*-periodic for* $k = -4$ *or* $k = 1$*;*
c) $\{B^*_{2n+1}(k)\}$ *is* 6*-periodic if* $k = -3$ *or* $k = 0$*;*
d) $\{B^*_{2n+1}(k)\}$ *is* 2*-periodic if* $k = -2$ *or* $k = -1$*.*

12.2. **Values of $x \in \frac{1}{2} + \mathbb{Z}$.** The example $x = -\frac{3}{2}$ is considered first.

Proposition 12.13. *For* $n \in \mathbb{N} \cup \{0\}$,

$$B^*_{2n+1}\left(-\tfrac{3}{2}\right) = 0. \tag{12.21}$$

Proof. Theorem 11.1 states that $B^*_{2n+1}(-x-3) = -B^*_{2n+1}(x)$. Replacing $x = -\frac{3}{2}$ gives the result. □

The symmetry of $B^*_{2n+1}(x)$ about $x = -\frac{3}{2}$ shows that it suffices to consider values of the form $k + \frac{1}{2}$ for $k \geq -1$.

Theorem 12.14. *For all* $k \geq -1$,

$$B^*_{2n+1}\left(k + \tfrac{1}{2}\right) = \frac{1}{2}\sum_{r=0}^{k+1} U_{2n}\left(\frac{2r+1}{4}\right). \tag{12.22}$$

Proof. The proof is similar to that of Theorem 12.6, so the details are omitted. □

The next lemma produces an unbounded value in the sum (12.22) when $k \geq 1$.

Lemma 12.15. *For* $n \in \mathbb{N}$

$$U_{2n}\left(\tfrac{5}{4}\right) = \frac{2^{2n+2} - 2^{-2n}}{3}. \tag{12.23}$$

Proof. This comes directly from (9.7). □

The next examples deal with values of $B^*_{2n+1}(k + \frac{1}{2})$ that do not contain the unbounded term $U_{2n}\left(\frac{5}{4}\right)$.

Lemma 12.16. *The sequence* $B^*_{2n+1}\left(-\frac{1}{2}\right)$ *is not periodic.*

Proof. Theorem 12.14, with $k = -1$, and (9.2) give

$$B^*_{2n+1}\left(-\tfrac{1}{2}\right) = \tfrac{1}{2}U_{2n}\left(\tfrac{1}{4}\right) = \frac{2}{\sqrt{15}}\sin\left((2n+1)\theta\right),$$

with $\cos\theta = \frac{1}{4}$. It follows from here that $\{B^*_{2n+1}(-\frac{1}{2})\}$ is not periodic. Indeed, if p were a period, then $B^*_{2n+2p+1}\left(-\frac{1}{2}\right) = B^*_{2n+1}\left(-\frac{1}{2}\right)$ implies

$$\tan((2n+1)\theta) = \cot p\theta \text{ for all } n \in \mathbb{N}. \tag{12.24}$$

Thus 3θ and θ must differ by an integer multiple of π; that is $2\theta = \pi m$. This is impossible if $\cos\theta = \frac{1}{4}$. □

Lemma 12.17. *The sequence $B^*_{2n+1}\left(\frac{1}{2}\right)$ is not periodic.*

Proof. In the case $k = 0$, Theorem 12.14 gives

$$B^*_{2n+1}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\left[U_{2n}\left(\tfrac{1}{4}\right) + U_{2n}\left(\tfrac{3}{4}\right)\right]. \tag{12.25}$$

To check that this is not a periodic sequence, use (9.5) to produce

$$\sum_{n=0}^{\infty}\left[U_{2n}\left(\tfrac{1}{4}\right) + U_{2n}\left(\tfrac{3}{4}\right)\right]t^n = \frac{8(1+t)(4t^2+3t+4)}{16t^4+24t^3+25t^2+24t+16}. \tag{12.26}$$

Periodicity of $B^*_{2n+1}\left(\frac{1}{2}\right)$ implies that the poles of of the right-hand side in (12.26) must be roots of a polynomial of the form $1 - t^p$. In particular, the arguments of these poles must be rational multiples of π. One of these poles is $t_0 = (1+3\sqrt{7}i)/8$, with argument $\alpha = \cos^{-1}\left(\frac{1}{8}\right)$. Therefore α must be a rational multiple of π. To obtain a contradiction, observe that

$$\omega_{m,n} := 2\cos\left(\frac{\pi m}{n}\right) \tag{12.27}$$

is a root of the monic polynomial $2T_n(x/2)$. It follows that $\omega_{m,n}$ is an algebraic integer and a rational number (namely $\frac{1}{4}$). This implies that it must be an integer (see [17, page 50]). This is a contradiction. □

These results are summarized in the next theorem.

Theorem 12.18. *There is no integer value of $k \neq -2$ for which $\{B^*_{2n+1}(k+\frac{1}{2})\}$ is periodic.*

Special values of $B^*_{2n}(x)$. The second case considered here deals with values of the subsequence $B^*_{2n}(x)$. Symbolic experiments were unable to produce nice closed-forms for special values of $B^*_{2n}(x)$, but the identity

$$B^*_{2n}(-1) = B^*_{2n}(-2) \tag{12.28}$$

motivated the definition of the function

$$A^*_{2n}(u) := B^*_{2n}(-1-u) - B^*_{2n}(-1), \text{ for } u \in \mathbb{Z}. \tag{12.29}$$

Lemma 12.19. *For $n \in \mathbb{N}$, the function $A^*_{2n}(u)$ satisfies $A^*_{2n}(u) = A^*_{2n}(1-u)$. Therefore, it suffices to describe $A^*_{2n}(u)$ for $u \geq 1$.*

Proof. This follows directly from Theorem 11.1. □

The next statement expresses the function A^*_{2n} in terms of the Chebsyshev polynomials of the second kind $U_{2n-1}(x)$.

Proposition 12.20. *The function A_{2n}^* is given by*

$$A_{2n}^*(u) = \frac{1}{2}\sum_{j=2}^{u+1} U_{2n-1}\left(\frac{u+1-j}{2}\right). \tag{12.30}$$

Proof. Iterate the identity (10.8). □

The expression in (5.6) yields the next result.

Lemma 12.21. *The generating function of the sequence $\{B_{2n}^*(x) - B_{2n}^*(-1)\}$ satisfies*

$$4\sum_{n=1}^{\infty} \left[B_{2n}^*(x) - B_{2n}^*(-1)\right] z^{2n} = -\psi(w-1-x) - \psi(w+2+x) + 2\psi(w) + \frac{1}{w}$$

with $w = z + 1/z$.

The proof of the next result is similar to that of Theorem 12.4.

Corollary 12.22. *The generating function*

$$\sum_{n=1}^{\infty} \left[B_{2n}^*(x) - B_{2n}^*(-1)\right] z^n \tag{12.31}$$

is a rational function of z if and only if $2x \in \mathbb{Z}$.

The next statement is an analogue of Theorems 12.12 and 12.18.

Theorem 12.23. *Let $A_{2n}^*(x) = B_{2n}^*(-1-x) - B_{2n}^*(-1)$ as above. Then*
1) The sequences $A_{2n}^*(1)$ and $A_{2n}^*(0)$ vanish identically.
2) The sequences $A_{2n}^*(2)$ and $A_{2n}^*(-1)$ are periodic with period 3. The repeating values are $\{\frac{1}{2}, -\frac{1}{2}, 0\}$.
3) The sequences $A_{2n}^*(3)$ and $A_{2n}^*(-2)$ grow linearly in n. Moreover, $A_{2n}^*(3) - n$ and $A_{2n}^*(-2) - n$ are periodic with period 3. The repeating values are $\{\frac{1}{2}, -\frac{1}{2}, 0\}$.
4) The sequence $A_{2n}^*(x)$ is unbounded for $x \geq 4$ and $x \leq -3$.

13. Additional properties of the Zagier polynomials

The Zagier polynomials $B_n^*(x)$ have a variety of interesting properties. These are recorded here for future studies.

Coefficients. The Zagier polynomial $B_n^*(x)$ has rational coefficients, some of which are integers. Figure 2 shows the number of *integer* coefficients in $B_n^*(x)$ as a function of n. The minimum values seems to occur at the powers 2^j, where the number of integer coeffcients is $j-1$.

Signs of coefficients and shifts. The coefficients of $B_n^*(x)$ do not have a fixed sign, but there is a tendency towards positivity. Figure 4 shows the excess of positive coefficients divided by the total number. On the other hand, the shifted polynomial $B_n^*(x+\frac{3}{2})$ appears to have only positive coefficients. The coefficients of the shifted polynomial appears to be *logconcave*. This notion is defined in terms of the operator $\mathcal{L}$ acting on sequences $\{a_j\}$ via $\mathcal{L}(\{a_j\}) = \{a_j^2 - a_{j-1}a_{j+1}\}$. A sequence is called logconcave if $\mathcal{L}(\{a_j\})$ is nonnegative. The sequence is called *infinitely logconcave* if any application of $\mathcal{L}$ produces positive sequences. The data suggests that the coefficients of $B_n^*(x+\frac{3}{2})$ form an infinitely logconcave sequence.

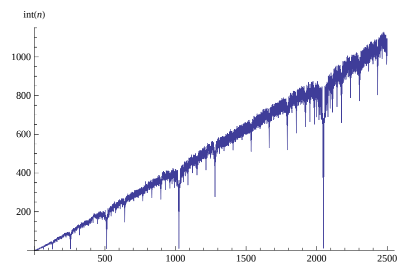

FIGURE 2. Integer coefficients

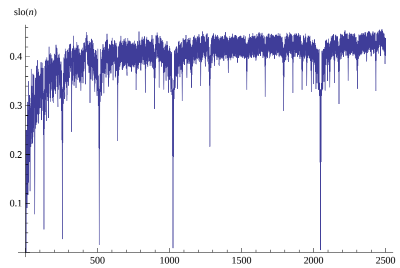

FIGURE 3. Linear behavior

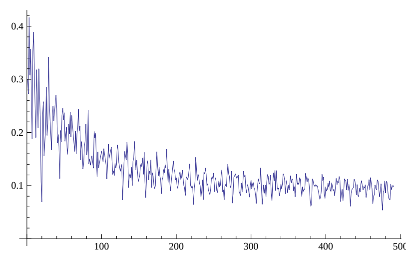

FIGURE 4. Excess of positive coefficients

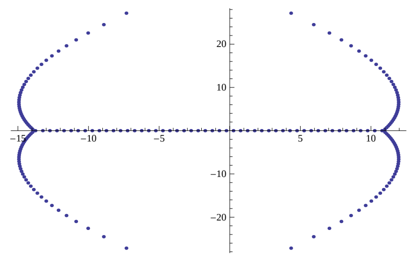

FIGURE 5. Roots of $B^*_{200}(x)$.

Roots of B^*_n. There is a well-established connection between the nature of the roots of a polynomials and the logconcavity of its coefficients. P. Brändén [3] has shown that if a polynomial has only real and negative roots, then its sequence of coefficients is infinitely logconcave. This motivated our computations of the roots of $B^*_n(x)$. The conclusion is that the polynomial $B^*_n\left(x+\frac{3}{2}\right)$ does not fall in this category and Brändén's criteria does not apply. Figure 5 shows these roots for $n = 200$.

A second shift. The polynomial $B^*_n(x-\frac{3}{2})$ admits a representation in terms of classical special functions. The Gegenbauer polynomial is defined by (see [18], p. 152, (6.37)):

$$C_n^{(\lambda)}(x) = \binom{n+2\lambda-1}{n} {}_2F_1\left(-n,\, n+2\lambda;\, \lambda+\tfrac{1}{2};\, \tfrac{1}{2}(1-x)\right). \tag{13.1}$$

Theorem 13.1. *The shifted Bernoulli polynomial $\tilde{B}_n(x)$, defined by $B^*_n\left(x-\frac{3}{2}\right)$ is given by*

$$\tilde{B}_n(x) = \frac{1}{n}T_n\left(\frac{x}{2}\right) + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B_{2k}(1/2)}{k2^{2k+2}} C_{n-2k}^{(2k)}\left(\frac{x}{2}\right). \tag{13.2}$$

Proof. Lemma 9.3 and expanding as a Taylor sum gives

$$\begin{aligned} \tilde{B}_n(x) &= \mathbb{E}\left[\frac{1}{n}T_n\left(\frac{x}{2}+\frac{1}{2}iL_B\right)\right] \\ &= \frac{1}{n}\left(T_n\left(\frac{x}{2}\right) + \sum_{k=1}^{n}\frac{1}{k!}\mathbb{E}\left[\left(\frac{iL_B}{2}\right)^k\right]\left(\frac{d}{dx}\right)^k T_n\left(\frac{x}{2}\right)\right). \end{aligned}$$

The hypergeometric representation of the Chebyshev polynomial

$$T_n(x) = {}_2F_1\left(n,\, -n;\, \tfrac{1}{2};\, \tfrac{1-x}{2}\right) \tag{13.3}$$

and the differentiation rule (Exercise 5.1 in [18], p. 128)

$$\left(\frac{d}{dx}\right)^k {}_2F_1\left(a,\, b;\, c;\, z\right) = \frac{(a)_k (b)_k}{(c)_k} {}_2F_1\left(a+k,\, b+k;\, c+k;\, z\right) \tag{13.4}$$

give the identity

$$\left(\frac{d}{dx}\right)^k T_n(x) = n2^{k-1}(k-1)! C_{n-k}^{(k)}(x). \tag{13.5}$$

The odd moments of L_B vanish and the even moments are given by

$$\mathbb{E}\left[\left(\frac{iL_B}{2}\right)^{2k}\right] = \frac{B_{2k}(\frac{1}{2})}{2^{2k}} \tag{13.6}$$

according to (2.19). □

The Chebyshev polynomial T_n and the Gegenbauer polynomial $C_{n-2k}^{(2k)}$ have the same parity as n. Thus Theorem 13.1 yields a new proof of Theorem 11.1, stated below in terms of $\tilde{B}_n$.

Corollary 13.2. *The shifted polynomials $\tilde{B}_n(x)$ have the same parity as n:*

$$\tilde{B}_n(-x) = (-1)^n \tilde{B}_n(x). \tag{13.7}$$

14. The Euler case

This section describes a parallel treatment of the Euler polynomial $E_n(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{tx}}{e^t+1}, \tag{14.1}$$

Their umbrae is

$$\text{eval}\left\{\exp(t\mathfrak{E}(x))\right\} = \frac{2e^{tx}}{e^t+1}. \tag{14.2}$$

The Euler numbers are defined by

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2e^t}{e^{2t}+1}, \tag{14.3}$$

and they appear as

$$E_n = 2^n E_n\left(\tfrac{1}{2}\right). \tag{14.4}$$

Their umbra $\mathfrak{E}$ is

$$\text{eval}(\exp(z\mathfrak{E})) = \text{sech}(z) \tag{14.5}$$

and the Euler numbers are expressed as

$$E_n = 2^n\left(\mathfrak{E} + \tfrac{1}{2}\right)^n, \tag{14.6}$$

which is an umbral equivalent of (14.4).

The next statement is the analogue of Theorem 2.3.

Theorem 14.1. *There exists a real valued random variable L_E with probability density $f_{L_E}(x)$ such that, for all admissible functions h,*

$$\text{eval}\{h(\mathfrak{E}(x))\} = \mathbb{E}\left[h(x - 1/2 + iL_E)\right] \tag{14.7}$$

where the expectation is defined in (2.12)*. The density of L_E is given by*

$$f_{L_E}(x) = \text{sech}(\pi x), \quad \textit{for } x \in \mathbb{R}. \tag{14.8}$$

In particular,

$$\text{eval}\{\exp(\mathfrak{E}(x))\} = \mathbb{E}\left[it(x - 1/2 + iL_E)\right] \tag{14.9}$$

and

$$E_n(x) = \mathbb{E}\left[(x - \tfrac{1}{2} + iL_E)^n\right]. \tag{14.10}$$

Proof. The proof is similar to the Bernoulli case in Theorem 2.3. In this case, entry 3.981.3 of [7]:

$$\int_0^\infty \text{sech}(ax)\cos(xt)\,dx = \frac{\pi}{2a}\text{sech}\left(\frac{\pi t}{2a}\right) \tag{14.11}$$

is employed. □

Note 14.2. The analogue of Example 2.5 is

$$\text{eval}\left\{\log \mathfrak{E}(x)\right\} = \log 2 + 2\log\Gamma\left(\tfrac{x+1}{2}\right) - 2\log\Gamma\left(\tfrac{x}{2}\right), \quad \text{for } x > \tfrac{1}{2} \tag{14.12}$$

and differentiation produces

$$\text{eval}\left\{\mathfrak{E}^{-k}(x)\right\} = \frac{(-1)^{k-1}}{(k-1)!}2\beta^{(k-1)}(x), \quad \text{for } x > \tfrac{1}{2}, \tag{14.13}$$

with

$$\beta(x) = \frac{1}{2}\left(\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right) \tag{14.14}$$

the beta function on page 906 of [7]. The proofs of all these results are similar to those presented for the Bernoulli case.

It is natural to consider now the modified Euler numbers

$$E_n^* = \sum_{r=0}^{n} \binom{n+r}{2r}\frac{n}{n+r}E_r, \quad n > 0. \tag{14.15}$$

Symbolic experimentation suggested the next statement. The proof of the next statement follows the same ideas as in the Bernoulli case.

Theorem 14.3. *The odd subsequence of the modified Euler numbers $\{E_{2n+1}^*\}$ is a periodic sequence of period* 3*, with values* $\{1, -2, 1\}$*.*

Define the modified Euler polynomials by

$$E_n^*(x) = \sum_{r=0}^{n} \binom{n+r}{2r}\frac{n}{n+r}E_r(x). \tag{14.16}$$

Then the even-order numbers $E_{2n}^*(0)$ have period 12 with values

n mod 12	0	2	4	6	8	10
$E_n^*(0)$	1	0	-2	3	-2	0

The proof of this result follows the same steps as in the Bernoulli case.

The final statement in this section is the analogue of Theorem 11.1.

Theorem 14.4. *The modified Euler polynomials satisfy*

$$E_n^*(-x-3) = (-1)^n E_n^*(x). \tag{14.17}$$

15. The duplication formula for Zagier polynomials

The identity

$$B_k(mx) = m^{k-1} \sum_{k=0}^{m-1} B_k\left(x + \frac{k}{m}\right) \tag{15.1}$$

was given by J. L. Raabe in 1851. The special case $m = 2$ gives the duplication formula for Bernoulli polynomials

$$2B_k(2x) = 2^k B_k(x) + 2^k B_k\left(x + \tfrac{1}{2}\right). \tag{15.2}$$

Summing over k yields

$$2\sum_{k=0}^{n} \binom{n+k}{2k} \frac{B_k(2x)}{n+k} = \sum_{k=0}^{n} \binom{n+k}{2k} \frac{2^k B_k(x)}{n+k} + \sum_{k=0}^{n} \binom{n+k}{2k} \frac{2^k B_k\left(x+\frac{1}{2}\right)}{n+k}. \tag{15.3}$$

An umbral interpretation of this identity leads to a duplication formula for the Zagier polynomials. This result is expressed in terms of the umbral composition defined next.

Definition 15.1. Given two sequences of polynomials $P = \{P_n(x)\}$ and $Q = \{Q_n(x)\}$, their *umbral composition* is defined as

$$(P \circ Q)_n(x) = \sum_{k=0}^{n} p_{k,n} Q_k(x), \tag{15.4}$$

where $p_{k,n}$ is the coefficient of x^k in $P_n(x)$.

The use of umbral composition is clarified in the next lemma.

Lemma 15.2. *Let P and Q be polynomials and assume*

$$P_n(x) = \text{eval}\left\{(x+\mathfrak{P})^n\right\} \text{ and } Q_n(x) = \text{eval}\left\{(x+\mathfrak{Q})^n\right\}. \tag{15.5}$$

Then

$$(P \circ Q)_n(x) = \text{eval}\left\{(x+\mathfrak{P}+\mathfrak{Q})^n\right\}. \tag{15.6}$$

Proof. Denoting the relevant umbrae by a subindex, then

$$\begin{aligned} \text{eval}_{\mathfrak{P},\mathfrak{Q}}\left\{(x+\mathfrak{P}+\mathfrak{Q})^n\right\} &= \text{eval}_{\mathfrak{Q}}\left\{P_n(x+\mathfrak{Q})\right\} \\ &= \sum_{k=0}^{n} p_{k,n} Q_k(x) \\ &= (P \circ Q)_n(x), \end{aligned}$$

as claimed. □

Consider now the Bernoulli and Euler umbrae

$$\text{eval}\{\exp(t\mathfrak{B})\} = \frac{t}{e^t - 1} \text{ and eval}\{\exp(t\mathfrak{E})\} = \frac{2}{e^t+1} \tag{15.7}$$

given in (2.7) and (14.2), respectively. The identity

$$\text{eval}\{\exp(t\mathfrak{B})\} \times \text{eval}\{\exp(t\mathfrak{E})\} = \text{eval}\{\exp(2t\mathfrak{B})\} \tag{15.8}$$

is written (at the umbrae level) as

$$\mathfrak{B} + \mathfrak{E} = 2\mathfrak{B}. \tag{15.9}$$

The first summand on the right of (15.3) contains the term

$$\begin{aligned} 2^k B_k(x) &= \text{eval}\left\{2^k(x+\mathfrak{B})^k\right\} \\ &= \text{eval}\left\{(2x+2\mathfrak{B})^k\right\} \\ &= \text{eval}\left\{(2x+\mathfrak{B}+\mathfrak{E})^k\right\} \\ &= \text{eval}\left\{(\mathfrak{B}\circ\mathfrak{E})_k(2x)\right\}. \end{aligned}$$

Lemma 15.2 has been used in the last step. Similarly

$$2^k B_k\left(x+\tfrac{1}{2}\right) = \text{eval}\left\{(\mathfrak{B}\circ\mathfrak{E})_k(2x+1)\right\}.$$

Thus, (15.3) reads

$$2B_n^*(2x) = (B^*\circ E)_n(2x) + (B^*\circ E)_n(2x+1) \tag{15.10}$$

that can also be expressed in the form

$$2B_n^*(2x) = (B^*\circ E(x))_n(x) + \left(B^*\circ E\left(x+\tfrac{1}{2}\right)\right)_n\left(x+\tfrac{1}{2}\right), \tag{15.11}$$

that is an analogue of (15.2) for the Zagier polynomials.

Acknowledgments. The second author acknowledges the partial support of NSF-DMS 1112656. The first author is a post-doctoral fellow funded in part by the same grant. The authors wish to thank T. Amdeberhan for his valuable input into this paper.

References

[1] J. Blissard. Theory of generic equations. *Quart. J. Pure Appl. Math.*, 4:279–305, 1861.

[2] K. Boyadzhiev. A note on Bernoulli polynomials and solitons. *Jour. Nonlinear Math. Phys.*, 14:174–178, 2007.

[3] P. Brändén. Iterated sequences and the geometry of zeros. *J. Reine Angew. Math.*, 658:115–131, 2011.

[4] Y. A. Brychkov. *Handbook of Special Functions. Derivatives, Integrals, Series and Other Formulas.* Taylor and Francis, Boca Raton, Florida, 2008.

[5] A. Dixit, V. Moll, and C. Vignat. The Zagier modification of Bernoulli numbers and a polynomial extension. Part II. *Preprint*, 2012.

[6] I. Gessel. Applications of the classical umbral calculus. *Algebra Universalis*, 49:397–434, 2003.

[7] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[8] M. P. Grosset and A. P Veselov. Bernoulli numbers and solitions. *Jour. Nonlinear Math. Phys.*, 12:469–474, 2005.

[9] M. Kervaire and J. Milnor. Groups of homotopy spheres: I. *Ann. Math.*, 77:504–537, 1963.

[10] J. P. Levine. Lectures on groups of homotopy spheres. In A. Ranicki N. Levitt F. Quinn, editor, *Alegebraic and Geometric Topology. Lecture Notes in Mathematics,* 1126, pages 62–95. Springer, Berlin - Heidelberg - New York, 1983.

[11] K. MacMillan and J. Sondow. Proofs of power sum and binomial coefficient congruences via Pascal's identity. *Amer. Math. Monthly*, 118:549–551, 2011.

[12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[13] M. Petkovšek, H. Wilf, and D. Zeilberger. *A=B*. A. K. Peters, Ltd., 1st edition, 1996.

[14] P. Ribenboim. *Fermat's Last Theorem for Amateurs*. Springer-Verlag, New York, 1st edition, 1999.

[15] J. Riordan. *Combinatorial Identities*. Wiley, New York, 1st edition, 1968.

[16] J. Spanier and K. Oldham. *An atlas of functions*. Hemisphere Publishing Co., 1st edition, 1987.

[17] I. Stewart and D. Tall. *Algebraic Number Theory*. Chapman and Hall, London, 1st edition, 1979.

[18] N. M. Temme. *Special Functions. An introduction to the Classical Functions of Mathematical Physics*. John Wiley and sons, New York, 1996.

[19] J. Touchard. Nombres exponentieles et nombres de Bernoulli. *Canad. J. Math.*, 8:305–320, 1956.

[20] Y. P. Yu. Bernoulli operator and Riemann's Zeta function. *ArXiv: math-NT/1011.3352×v3, 19541*, 2012.

[21] D. Zagier. A modified Bernoulli number. *Nieuw Archief voor Wiskunde*, 16:63–72, 1998.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: adixit@tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118 and L.S.S. Supelec, Universite d'Orsay, France
E-mail address: vignat@tulane.edu