Honors Calculus 131. Practice Test 2. Due October 29.

1) Use the mean value theorem to prove the inequality

$$|\sin a - \sin b| \leq |a - b|.$$

2) Sketch the graph of the function $f(x) = (x-1)^2(x+2)^3$.

3) Show that the function $f(x) = (1+x)/(1+x^2)$ has three inflection points and that they all lie on a straight line.

4) Prove that $g(x) = x|x|$ has an inflection point at $x = 0$, but $g''(0)$ does not exist.

5) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

6) A box with a square base and open top must have a volume of $32,000\text{cm}^2$. Find the dimensions of the box that minimizes the amount of material used.

7) Use Newton's method to approximate the positive root of $\sin x = x^2$. How many iterations do you need to ensure that you know the root within 10^{-3}?

8) Find the equation of the line that is tangent to the curve $y = -\sin x$, passes through the origin and has the maximum possible slope. You may need to use Newton's method to compute the answer.

9) Prove that the equation $3x + 2\cos x + 5 = 0$ has exactly one root.

10) Show that, for $x > 0$, we have

$$\frac{x}{1+x^2} < \tan^{-1} x < x.$$

11) Find the exact area under the curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leq x \leq \pi/2$ by using a Riemann sum.

12) Prove that

$$\frac{\sqrt{2}\pi}{24} < \int_{\pi/6}^{\pi/4} \cos x \, dx < \frac{\sqrt{3}\,\pi}{24}.$$

13) Evaluate

$$\lim_{n\to\infty} n \sum_{j=1}^{n} \frac{1}{n^2 + j^2}.$$

14) Evaluate the integral of x^{-2} between 1 and 2 by choosing an appropriate partition and then approximating the integral on (x_{i-1}, x_i) by the rectangle centered at $\sqrt{x_{i-1}x_i}$.

15) Find the derivative of the function

$$f(x) = \int_{\sin x}^{2x} \frac{u^2 - 1}{u^2 + 3} \, du.$$

16) Find a number a and a function f such that

$$6 + \int_a^x \frac{f(t)}{t^2} \, dt = 2\sqrt{x}.$$

17) Sketch the graph of the function $f(x) = x^4 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Discuss its critical points.

18) Prove that if $f'(a) > 0$ and f' is continuous at $x = a$ then there is an interval containing a where f is increasing. Compare this result with the example $f(x) = x/2 + x^2 \sin 1/x$.

19) Prove that if f is a continuous function on the bounded interval $[a, b]$, then the integral of f over $[a, b]$ exists and is finite. **Hint**. Compare the upper and lower Riemann sums.

20) Let f be a continuous function with a inverse g. Prove the identity

$$\int_a^b g(y)\,dy = bg(b) - ag(a) - \int_{g(a)}^{g(b)} f(x)\,dx.$$

Hint. From a partition of $[a, b]$ create a partition of the corresponding interval in the x-axis.

21) Use the previous result to evaluate

$$\int_a^b x^{1/n}\,dx.$$

22) Suppose that f is a continuous increasing function with $f(0) = 0$. Prove that for $a, b > 0$ we have *Young's inequality*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx,$$

and that equality holds if and only if $b = f(a)$. bf Hint. Draw a picture.

23) Find all continuous functions f that satisfy

$$\int_0^x f(t)\,dt = f^2(x) + C$$