Honors Calculus 131. Problem set 6.
Due October 21.

1) Prove that

$$\int_0^b x^3\,dx = \frac{1}{4}b^4,$$

by computing the lower and upper Riemann sums corresponding to equally spaced point on the interval $[0,\, b]$.

2) Compute the integral of $f(x) = x^p$ from a to b by using a partiton of the interval $[a,\, b]$ of the form $P = \{x_0,\, x_1,\, \cdots, x_n\}$ designed in such a way that the *ratio* $r = x_i/x_{i-1}$ is constant. Observe that normally one chooses a partition with the *difference* $x_i - x_{i-1}$ constant (then we simply say that the points are equally spaced).

3) Here is another way to solve problem 2). Define

$$c_n := \int_0^1 x^p\,dx.$$

Prove that

$$\int_0^a x^p\,dx = c_n a^{n+1}.$$

Check also that

$$\int_0^{2a} x^p\,dx = \int_{-a}^{a} (x+a)^p\,dx.$$

Use the binomial theorem to check that

$$2^{p+1} c_p a^{p+1} = 2a^{p+1} \sum_{k \text{ even}} \binom{p}{k} c_k.$$

Confirm the result by mathematical induction.

4) Compute the derivative of the function

$$F(x) = \int_0^x x f(t)\,dt.$$

Use the previous result to check that

$$\int_0^x \left(\int_0^u f(t)\,dt \right) du = \int_0^x (x-u) f(u)\,du.$$

5) Find $\left(f^{-1}\right)'(0)$ if

$$f(x) = \int_0^x \left[1 + \sin(\sin t)\right]\,dt.$$