THE EXPANSION OF SOME CLASSICAL FUNCTION INVOLVING BERNOULLI NUMBERS

The **Bernoulli numbers** defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \tag{1}$$

also appear in some other functions that are simple modifications of (1).

The first one comes by multiplying the generating function by a fixed power. This gives

$$\begin{aligned} \frac{t^m}{e^t - 1} &= t^m \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} B_k \frac{t^{k+m}}{k!} \\ &= \sum_{k=m}^{\infty} B_{k-m} \frac{t^k}{(k-m)!} \end{aligned} \tag{2}$$

Now introduce the notation

$$[\,t^r\,]\, f(t) = \text{Coefficient of } t^r \text{ in the expansion of } f(t). \tag{3}$$

Then (2) states that

$$\left[\,t^k\right] \frac{t^m}{e^t - 1} = \begin{cases} 0 & \text{if } 0 \leq k < m \\ B_{k-m}/(k-m)! & \text{if } k \geq m. \end{cases} \tag{4}$$

To obtain another function, write (1) in the form

$$\frac{1}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^{k-1}}{k!} \tag{5}$$

and the observe that

$$\frac{e^t + 1}{e^t - 1} = \frac{e^t - 1 + 2}{e^t - 1} = 1 + \frac{2}{e^t - 1} \tag{6}$$

and this can be written as

$$\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} = 1 + \frac{2}{e^t - 1}. \tag{7}$$

The function on the left is the quotient of hyperbolic cosine over hyperbolic sine, so it is the *hyperbolic cotangent*

$$\coth \frac{t}{2} = 1 + \frac{2}{e^t - 1}. \tag{8}$$

The expansion on the right of (8) is

$$1+2\sum_{k=0}^{\infty}\frac{B_k}{k!}t^{k-1}. \tag{9}$$

The first three terms on the right are

$$2B_0t^{-1}+(2B_1+1)+\frac{2B_2}{2!}t=\frac{2}{t}+\frac{1}{6}t \tag{10}$$

The identity

$$\coth\frac{t}{2}=1+2\sum_{k=0}^{\infty}\frac{B_k}{k!}t^{k-1}, \tag{11}$$

after replacing t by $2t$ becomes

$$\begin{aligned}\coth t &= 1+\sum_{k=0}^{\infty}\frac{B_k}{k!}2^kt^{k-1}\\ &= \frac{1}{t}+\frac{t}{3}+\sum_{k=2}^{\infty}\frac{2^{k+1}B_{k+1}}{(k+1)!}t^k\end{aligned} \tag{12}$$

The Bernoulli numbers appearing in the last series vanish when $k+1$ is odd, that is, k is even. Therefore, we let $k=2j+1$ and the new index j starts at $j=1$. This gives

$$\begin{aligned}\coth t &= \frac{1}{t}+\frac{t}{3}+\sum_{j=1}^{\infty}\frac{2^{2j+2}B_{2j+2}}{(2j+2)!}t^{2j+1}\\ &= \frac{1}{t}+\frac{t}{3}+\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}t^{2j-1}\\ &= \sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}t^{2j-1}.\end{aligned} \tag{13}$$

Now observe that

$$\coth it=\frac{\cosh it}{\sinh it}=\frac{\cos t}{i\sin t}=-i\cot t \tag{14}$$

and replacing t by it in (13) gives, using

$$i^{2j-1}=(-1)^{j-1}i \tag{15}$$

gives

$$-i\cot t=\sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}(-1)^{j-1}it^{2j-1}.$$

Therefore

$$\cot t=\sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}(-1)^{j}t^{2j-1}.$$

Theorem 1. *The coefficients of* $\cot t$ *around* $t=0$ *are given by*

$$[t^j]\cot t=\begin{cases}0 & \textit{if } j \textit{ is even}\\ (-1)^i2^{2i}B_{2i}/(2i)! & \textit{if } j \textit{ is odd with } j=2i-1\end{cases}$$

Another function that can be written using the generating function of the Bernoulli numbers is

$$\frac{t}{\sin t} = \frac{2it}{e^{it} - e^{-it}} = \frac{2it}{e^{2it} - 1} \times e^{it} \tag{16}$$

The product is now expanded in series as

$$\frac{t}{\sin t} = \left(\sum_{k=0}^{\infty} (2i)^k B_k \frac{t^k}{k!} \right) \times \left(\sum_{j=0}^{\infty} i^j \frac{t^j}{j!} \right) \tag{17}$$

and using the formula to multiply series

$$\left(\sum_{j=0}^{\infty} a_j \frac{t^j}{j!} \right) \times \left(\sum_{k=0}^{\infty} b_k \frac{t^k}{k!} \right) = \sum_{r=0}^{\infty} \left[\sum_{\ell=0}^{r} \binom{r}{\ell} a_{r-\ell} b_\ell \right] \frac{t^r}{r!} \tag{18}$$

gives

$$\frac{t}{\sin t} = \sum_{r=0}^{\infty} \left[i^r \sum_{k=0}^{r} \binom{r}{k} B_k 2^k \right] \frac{t^r}{r!} \tag{19}$$

The fact that the left-hand side is an even function shows that

$$\sum_{k=0}^{r} \binom{r}{k} B_k 2^k = 0 \quad \text{for } r \text{ odd.} \tag{20}$$

Another approach to the is given next:

$$\sum_{k=0}^{r} \binom{r}{k} B_k 2^k = 2^r \sum_{k=0}^{r} \binom{r}{k} B_k \left(\frac{1}{2} \right)^{r-k} \tag{21}$$

Using (18) we see that the sum is the coefficient of $t^r/r!$ in the product

$$\begin{aligned} \left(\sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \right) \times \left(\sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k \frac{t^k}{k!} \right) &= \frac{t}{e^t - 1} \times e^{t/2} \\ &= \frac{t}{e^{t/2} - e^{-t/2}} \\ &= \frac{t/2}{\sinh(t/2)} \end{aligned} \tag{22}$$

and this function is even, so the odd coefficients must vanish. *This is too long* and we are still using the fact that some function is even. Can one prove this directly?

Up to now we have

$$\begin{aligned} \frac{t}{\sin t} &= \sum_{r=0}^{\infty} \left[(-1)^r \sum_{k=0}^{2r} \binom{2r}{k} B_k 2^k \right] \frac{t^{2r}}{(2r)!} \\ &= \sum_{r=0}^{\infty} \left[(-1)^r \left(-2r + \sum_{k=0}^{r} \binom{2r}{2k} B_{2k} 2^{2k} \right) \right] \frac{t^{2r}}{(2r)!} \end{aligned} \tag{23}$$

This proves:

$$[\,t^{2r}\,]\frac{t}{\sin t} = \frac{(-1)^r}{(2r)!}\left(-2r + \sum_{k=0}^{r}\binom{2r}{2k}B_{2k}2^{2k}\right) \tag{24}$$

Question: is there a way to simplify this?

The basic identity

$$\cot x - 2\cot 2x = \tan x \tag{25}$$

can be used to compute the *tangent numbers* defined by the expansion

$$\tan x = \sum_{n=0}^{\infty} T_n \frac{x^n}{n!}. \tag{26}$$

The result of Theorem 1 shows that $T_n = 0$ for n even. This is consistent with the fact that $\tan x$ is an odd function. For n odd, say $n = 2j-1$, the coefficient of t^n on the left-hand side is also obtained from Theorem 1:

$$\begin{aligned}
[\,x^{2j-1}\,]\,(\cot x - 2\cot 2x) &= (-1)^j 2^{2j}\frac{B_{2j}}{(2j)!} - 2\cdot 2^{2j-1}\cdot(-1)^j 2^{2j}\frac{B_{2j}}{(2j)!} \\
&= (-1)^j 2^{2j}\frac{B_{2j}}{(2j)!}\left[1 - 2^{2j}\right] \\
&= (-1)^{j-1} 2^{2j}\frac{B_{2j}}{(2j)!}\left[2^{2j} - 1\right].
\end{aligned} \tag{27}$$

Theorem 2. *The tangent numbers, defined by the series*

$$\tan x = \sum_{k=0}^{\infty} T_n \frac{x^n}{n!} \tag{28}$$

are given by

$$T_{2n} = 0, \tag{29}$$

and

$$T_{2n-1} = (-1)^{n-1}2^{2n}\frac{B_{2n}}{(2n)!}\left[2^{2n} - 1\right]. \tag{30}$$