THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 14: THE EXPONENTIAL INTEGRAL FUNCTION. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 6, 2006.

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ABSTRACT. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated using the exponential integral function. Some examples are discussed.

1. Introduction

The *exponential integral* function is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt \tag{1.1}$$

for $x < 0$. In the case $x > 0$ we use the Cauchy principal value

$$\mathrm{Ei}(x) = -\lim_{\epsilon \to 0^+} \left[\int_{-x}^{-\epsilon} \frac{e^{-t}}{t}\, dt + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t}\, dt \right]. \tag{1.2}$$

• 3.351.6 This appears as 3.351.6.

Another function defined by an integral is the *logarithmic integral*:

$$\mathrm{li}(u) := \int_0^u \frac{dx}{\ln x}. \tag{1.3}$$

• 4.211.2 This appears as 4.211.2. The change of variables $t = \ln x$ shows that

$$\mathrm{li}(u) = \mathrm{Ei}(\ln u). \tag{1.4}$$

Observe that the integral defining li diverges as $u \to \infty$. Indeed, 4.211.1 states that

• 4.211.1

$$\int_e^{\infty} \frac{dx}{\ln x} = +\infty \tag{1.5}$$

This is evident from the change of variables $t = \ln x$ that yields

$$\int_e^{\infty} \frac{dx}{\ln x} = \int_1^{\infty} \frac{e^t\, dt}{t} \geq \int_1^{\infty} \frac{dt}{t} = \infty. \tag{1.6}$$

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2. Some simple changes of variables

The change of variables $t = -as$ yields

$$\int_{-x/a}^{\infty} \frac{e^{-as}}{s} ds = -\text{Ei}(x). \tag{2.1}$$

In particular, the choice $x = -a$ yields 3.351.5: • 3.351.5

$$\int_{1}^{\infty} \frac{e^{-as}}{s} ds = -\text{Ei}(-a). \tag{2.2}$$

3. Some logarithmic integrals

The exponential integral function Ei allows the evaluation of several logarithmic integrals. For instance 4.212.1: • 4.212.1

$$\int_0^1 \frac{dx}{a + \ln x} = e^{-a}\text{Ei}(a) \tag{3.1}$$

follows from the change of variables $t = a + \ln x$. Similarly, 4.212.2: • 4.212.2

$$\int_0^1 \frac{dx}{a - \ln x} = -e^{a}\text{Ei}(-a) \tag{3.2}$$

is evaluated using $t = a - \ln x$.

We now consider the family

$$f_n(a) := \int_0^1 \frac{dx}{(a + \ln x)^n}. \tag{3.3}$$

The change of variables $t = a + \ln x$ gives

$$f_n(a) = e^{-a} \int_{-\infty}^{a} t^{-n} e^t \, dt. \tag{3.4}$$

Integrate by parts to produce

$$\int_{-\infty}^{a} \frac{e^t \, dt}{t^n} = \frac{e^a a^{1-n}}{1-n} - \frac{1}{1-n} \int_{-\infty}^{a} \frac{e^t \, dt}{t^{n-1}}. \tag{3.5}$$

This yields a recurrence for the integrals $f_n(a)$:

$$f_n(a) = -\frac{a^{1-n}}{n-1} + \frac{1}{n-1} f_{n-1}(a). \tag{3.6}$$

The initial value is given in 4.212.1. From here we deduce and prove by induction, formula 4.212.8: • 4.212.8

$$\int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{e^{-a}}{(n-1)!}\text{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(n-k-1)!}{a^{n-k}}. \tag{3.7}$$

Using (3.4) we obtain 3.351.4: • 3.351.4

$$\int_a^{\infty} \frac{e^{-px} \, dx}{x^{n+1}} = \frac{(-1)^{n+1} p^n}{n!}\text{Ei}(-ap) + \frac{e^{-ap}}{a^n n!} \sum_{k=0}^{n-1} (-1)^k p^k a^k (n-k-1)! \tag{3.8}$$

The integral 4.212.3: • 4.212.3

$$\int_0^1 \frac{dx}{(a + \ln x)^2} = -\frac{1}{a} + e^{-a}\text{Ei}(a) \tag{3.9}$$

• 4.212.5 is the special case $n = 2$ of (3.7). The integral 4.212.5:

$$\int_0^1 \frac{\ln x \, dx}{(a + \ln x)^2} = 1 + (1 - a)e^{-a}\text{Ei}(a) \tag{3.10}$$

can be obtained from

$$\frac{\ln x}{(a + \ln x)^2} = \frac{1}{a + \ln x} - \frac{a}{(a + \ln x)^2}. \tag{3.11}$$

• 4.212.9 Similar arguments produce 4.212.9:

$$\int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{(-1)^n e^a \text{Ei}(-a)}{(n-1)!} + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)!(-a)^{k-n}. \tag{3.12}$$

• 4.212.4 The formula 4.212.4:

$$\int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^a \text{Ei}(-a) \tag{3.13}$$

is the special case $n = 2$. Writing

$$\ln x = a - (a - \ln x) \tag{3.14}$$

• 4.212.6 we obtain the evaluation of 4.212.6:

$$\int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1 + a)e^a \text{Ei}(-a). \tag{3.15}$$

4. The exponential scale

The change of variables $t = -ae^{nu}$ produces

$$\text{Ei}(x) = -n \int_c^\infty \exp\left(-ae^{nu}\right) \, du, \tag{4.1}$$

where $c = \frac{1}{n} \ln(-x/a)$. The choice $x = -a$ produces

$$\text{Ei}(-a) = -n \int_0^\infty \exp\left(-ae^{nu}\right). \tag{4.2}$$

• 3.327 This appears as 3.327 in [1].

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References

[1] I.S. Gradshteyn and I.M. Rhyzik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 6th edition, 2000.

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