

The integrals in Gradshteyn and Ryzhik. Part 2: Elementary logarithmic integrals

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ABSTRACT. We describe methods to evaluate elementary logarithmic integrals. The integrand is the product of a rational function and a linear polynomial in $\ln x$.

1. Introduction

The table of integrals by I. M. Gradshteyn and I. M. Ryzhik [3] contains a large selection of definite integrals of the form

$$(1.1) \quad \int_a^b R(x) \ln^m x \, dx,$$

where $R(x)$ is a rational function, $a, b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. We call integrals of the form (1.1) *elementary logarithmic integrals*. The goal of this note is to present methods to evaluate them. We may assume that $a = 0$ using

$$(1.2) \quad \int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \ln^m x \, dx - \int_0^a R(x) \ln^m x \, dx.$$

Section 2 describes the situation when R is a polynomial. Section 3 presents the case in which the rational function has a single simple pole. Finally section 4 considers the case of multiple poles.

2. Polynomials examples

The first example considered here is

$$(2.1) \quad I(P; b, m) := \int_0^b P(x) \ln^m x \, dx,$$

2000 *Mathematics Subject Classification*. Primary 33.

Key words and phrases. Integrals.

The author wishes to thank Luis Medina for a careful reading of an earlier version of the paper. The partial support of NSF-DMS 0409968 is also acknowledged.

where P is a polynomial. This can be evaluated in elementary terms. Indeed, $I(P; b, m)$ is a linear combination of

$$(2.2) \quad \int_0^b x^j \ln^m x \, dx,$$

and the change of variables $x = bt$ yields

$$(2.3) \quad \int_0^b x^j \ln^m x \, dx = b^{j+1} \sum_{k=0}^m \binom{m}{k} \ln^{m-k} b \int_0^1 t^j \ln^k t \, dt.$$

The last integral evaluates to $(-1)^k k! / (j+1)^{k+1}$ either an easy induction argument or by the change of variables $t = e^{-s}$ that gives it as a value of the gamma function.

Theorem 2.1. Let $P(x)$ be a polynomial given by

$$(2.4) \quad P(x) = \sum_{j=0}^p a_j x^j.$$

Then

$$(2.5) \quad I(P; b, m) := \int_0^b P(x) \ln^m x \, dx = \sum_{k=0}^m (-1)^k k! \binom{m}{k} \ln^{m-k} b \sum_{j=0}^p a_j \frac{b^{j+1}}{(j+1)^{k+1}}.$$

This expression shows that $I(P; b, m)$ is a linear combination of $b^j \ln^k b$, with $1 \leq j \leq 1+p (= 1 + \deg(P))$ and $0 \leq k \leq m$.

3. Linear denominators

We now consider the integral

$$(3.1) \quad f(b; r) := \int_0^b \frac{\ln x \, dx}{x+r}$$

for $b, r > 0$. This corresponds to the case in which the rational function in (1.1) has a single simple pole.

The change of variables $x = rt$ produces

$$(3.2) \quad \int_0^b \frac{\ln x \, dx}{x+r} = \ln r \ln(1 + b/r) + \int_0^{b/r} \frac{\ln t \, dt}{1+t}.$$

Therefore, it suffices to consider the function

$$(3.3) \quad g(b) := \int_0^b \frac{\ln t \, dt}{1+t},$$

as we have

$$(3.4) \quad f(b; r) = \ln r \ln \left(1 + \frac{b}{r} \right) + g \left(\frac{b}{r} \right).$$

Before we present a discussion of the function g , we describe some elementary consequences of (3.2).

Elementary examples. The special case $r = b$ in (3.2) yields

$$(3.5) \quad \int_0^b \frac{dx}{x+b} = \ln 2 \ln b + \int_0^1 \frac{\ln t dt}{1+t}.$$

Expanding $1/(1+t)$ as a geometric series, we obtain

$$(3.6) \quad \int_0^1 \frac{\ln t dt}{1+t} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}.$$

This appears as **4.231.1** in [3]. Differentiating (3.2) with respect to r produces

$$(3.7) \quad \int_0^b \frac{\ln x dx}{(x+r)^2} = -\frac{\ln(b+r)}{r} + \frac{\ln r}{r} + \frac{b \ln b}{r(r+b)}.$$

As $b, r \rightarrow 1$ we obtain

$$(3.8) \quad \int_0^1 \frac{\ln x dx}{(1+x)^2} = -\ln 2.$$

This appears as **4.231.6** in [3]. On the other hand, as $b \rightarrow \infty$ we recover **4.231.5** in [3]:

$$(3.9) \quad \int_0^\infty \frac{\ln x dx}{(x+r)^2} = \frac{\ln r}{r}.$$

The polylogarithm function. The evaluation of the integral

$$(3.10) \quad g(b) := \int_0^b \frac{\ln t dt}{1+t},$$

requires the transcendental function

$$(3.11) \quad \text{Li}_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

This is the *polylogarithm function* and it has also appeared in [5] in our discussion of the family

$$(3.12) \quad h_n(a) := \int_0^\infty \frac{\ln^n x dx}{(x-1)(x+a)}, \quad n \in \mathbb{R}, a > 0.$$

In the current context we have $n = 2$ and we are dealing with the *dilogarithm function*: $\text{Li}_2(x)$.

Lemma 3.1. The function $g(b)$ is given by

$$(3.13) \quad g(b) = \ln b \ln(1+b) + \text{Li}_2(-b).$$

PROOF. The change of variables $t = bs$ yields

$$(3.14) \quad g(b) = \ln b \ln(1+b) + \int_0^1 \frac{\ln s ds}{1+bs}.$$

Expanding the integrand in a geometric series yields the final identity. □

Theorem 3.2. Let $b, r > 0$. Then

$$(3.15) \quad \int_0^b \frac{\ln x \, dx}{x+r} = \ln b \ln \left(\frac{b+r}{r} \right) + \text{Li}_2 \left(-\frac{b}{r} \right).$$

Corollary 3.3. Let $b > 0$. Then

$$(3.16) \quad \int_0^b \frac{\ln x \, dx}{x+b} = \ln 2 \ln b - \frac{\pi^2}{12}.$$

PROOF. Let $r \rightarrow b$ in Theorem 3.2 and use

$$(3.17) \quad \text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

□

The expression in Theorem 3.2 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:

Corollary 3.4. Let $0 < a < b$ and $r_1 \neq r_2 \in \mathbb{R}^+$. Then, with $r = r_2 - r_1$, we have

$$\begin{aligned} \int_a^b \frac{\ln x \, dx}{(x+r_1)(x+r_2)} &= \frac{1}{r} \left[\ln b \ln \left(\frac{r_2(b+r_1)}{r_1(b+r_2)} \right) + \ln a \ln \left(\frac{r_1(a+r_2)}{r_2(a+r_1)} \right) \right] + \\ &+ \frac{1}{r} \left[\text{Li}_2 \left(-\frac{b}{r_1} \right) - \text{Li}_2 \left(-\frac{a}{r_1} \right) - \text{Li}_2 \left(-\frac{b}{r_2} \right) + \text{Li}_2 \left(-\frac{a}{r_2} \right) \right]. \end{aligned}$$

The special case $a = r_1$ and $b = r_2$ is of interest:

Corollary 3.5. Let $0 < a < b$. Then

$$\begin{aligned} \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} &= \frac{1}{b-a} [\ln(ab) \ln(a+b) - \ln 2 \ln(ab) - 2 \ln a \ln b] \\ &+ \frac{1}{b-a} \left[-2\text{Li}_2(-1) + \text{Li}_2 \left(-\frac{b}{a} \right) + \text{Li}_2 \left(-\frac{a}{b} \right) \right]. \end{aligned}$$

The integral in Corollary 3.5 appears as **4.232.1** in [3]. An interesting problem is to derive **4.232.2**

$$(3.18) \quad \int_0^{\infty} \frac{\ln x \, dx}{(x+u)(x+v)} = \frac{\ln^2 u - \ln^2 v}{2(u-v)}$$

directly from Corollary 3.5.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that

$$(3.19) \quad \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} = \frac{\ln ab}{2(b-a)} \ln \frac{(a+b)^2}{4ab}.$$

PROOF. The partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right).$$

reduces the problem to the evaluation of

$$I_1 = \int_a^b \frac{\ln x \, dx}{x+a} \text{ and } I_2 = \int_a^b \frac{\ln x \, dx}{x+b}.$$

The change of variables $x = at$ gives, with $c = b/a$,

$$\begin{aligned} I_1 &= \int_1^c \frac{\ln(at) \, dt}{1+t} \\ &= \ln a \int_1^c \frac{dt}{1+t} + \int_1^c \frac{\ln t}{1+t} dt \\ &= \ln a \ln(1+c) - \ln a \ln 2 + \int_1^c \frac{\ln t}{1+t} dt. \end{aligned}$$

Similarly,

$$I_2 = \ln b \ln 2 - \ln b \ln(1+1/c) + \int_1^{1/c} \frac{\ln t}{1+t} dt.$$

Therefore

$$\begin{aligned} I_1 - I_2 &= \ln a \ln(1+c) + \ln b \ln(1+1/c) - \ln 2 \ln a - \ln 2 \ln b + \\ &+ \int_1^c \frac{\ln t}{1+t} dt - \int_{1/c}^1 \frac{\ln t}{1+t} dt. \end{aligned}$$

Let $s = 1/t$ in the second integral to get

$$\int_{1/c}^1 \frac{\ln t}{1+t} dt = \int_c^1 \frac{\ln s}{s(1+s)} ds.$$

Replacing in the expression for $I_1 - I_2$ yields

$$\begin{aligned} I_1 - I_2 &= \ln a (\ln(a+b) - \ln a - \ln 2) - \ln b (\ln 2 - \ln(a+b) + \ln b) + \\ &+ \int_1^c \frac{\ln t}{t} dt. \end{aligned}$$

The last integral can now be evaluated by elementary means to produce the result. \square

Now comparing the two evaluations of the integral in Corollary 3.5 produces an identity for the dilogarithm function.

Corollary 3.6. The dilogarithm function satisfies

$$(3.20) \quad \text{Li}_2(-z) + \text{Li}_2\left(-\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(z).$$

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [4] a nice description of them.

4. A single multiple pole

In this section we consider the evaluation of

$$(4.1) \quad f_n(b, r) := \int_0^b \frac{\ln x \, dx}{(x+r)^n}.$$

This corresponds to the elementary rational integrals with a single pole (at $x = -r$). The change of variables $x = rt$ yields

$$f_n(b, r) = \frac{\ln r}{(n-1)r^{n-1}} \left[\frac{(b+r)^{n-1} - r^{n-1}}{(b+r)^{n-1}} \right] + \frac{1}{r^{n-1}} h_n(b/r),$$

where

$$(4.2) \quad h_n(b) := \int_0^b \frac{\ln t \, dt}{(1+t)^n}.$$

We first establish a recurrence for h_n .

Theorem 4.1. Let $n > 2$ and $b > 0$. Then h_n satisfies the recurrence

$$(4.3) \quad h_n(b) = \frac{n-2}{n-1} h_{n-1}(b) + \frac{b \ln b}{(n-1)(1+b)^{n-1}} + \frac{1 - (1+b)^{n-2}}{(n-1)(n-2)(1+b)^{n-2}}.$$

PROOF. Start with

$$h_n(b) = \int_0^b \frac{[(1+t) - t] \ln t \, dt}{(1+t)^n} = h_{n-1}(b) - \int_0^b \frac{t \ln t \, dt}{(1+t)^n}.$$

Integrate by parts in the last integral, with $u = t \ln t$ and $dv = dt/(1+t)^n$ to produce the result. \square

The initial condition for this recurrence is obtained from the value

$$(4.4) \quad h_2(b) = \frac{b}{1+b} \ln b - \ln(1+b).$$

This expression follows by a direct integration by parts in

$$(4.5) \quad h_2(b) = -\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^b \ln t \frac{d}{dt} (1+t)^{-1} dt.$$

The first few values of $h_n(b)$ suggest the introduction of the function

$$(4.6) \quad q_n(b) := (1+b)^{n-1} h_n(b),$$

for $n \geq 2$. For example,

$$(4.7) \quad q_2(b) = b \ln b - (1+b) \ln(1+b).$$

The recurrence for h_n yields one for q_n .

Corollary 4.2. The recurrence

$$(4.8) \quad q_n(b) = \frac{(n-2)}{(n-1)} (1+b) q_{n-1}(b) + \frac{b \ln b}{n-1} - \frac{(1+b) [(1+b)^{n-2} - 1]}{(n-1)(n-2)},$$

holds for $n \geq 3$.

Corollary 4.2 establishes the existence of functions $X_n(b)$, $Y_n(b)$ and $Z_n(b)$, such that

$$(4.9) \quad q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1+b) + Z_n(b).$$

The recurrence (4.8) produces explicit expression for each of these parts.

Proposition 4.3. Let $n \geq 2$ and $b > 0$. Then

$$(4.10) \quad X_n(b) = \frac{(1+b)^{n-1} - 1}{n-1}.$$

PROOF. The function X_n satisfies the recurrence

$$(4.11) \quad X_n(b) = \frac{n-2}{n-1}(1+b)X_{n-1}(b) + \frac{b}{n-1}.$$

The initial condition is $X_2(b) = b$. The result is now easily established by induction. \square

Proposition 4.4. Let $n \geq 2$ and $b > 0$. Then

$$(4.12) \quad Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}.$$

PROOF. The function Y_n satisfies the recurrence

$$(4.13) \quad Y_n(b) = \frac{n-2}{n-1}(1+b)Y_{n-1}(b).$$

This recurrence and the initial condition $Y_2(b) = -(1+b)$, yield the result. \square

It remains to identify the function $Z_n(b)$. It satisfies the recurrence

$$(4.14) \quad Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)[(1+b)^{n-2} - 1]}{(n-2)(n-1)}.$$

This recurrence and the initial condition $Z_2(b) = 0$ suggest the definition

$$(4.15) \quad T_n(b) := -\frac{(n-1)!Z_n(b)}{b(1+b)}.$$

Lemma 4.5. The function $T_n(b)$ is a polynomial of degree $n-3$ with positive integer coefficients.

PROOF. The function $T_n(b)$ satisfies the recurrence

$$(4.16) \quad T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)! \left[\frac{(1+b)^{n-2} - 1}{b} \right].$$

Now simply observe that the right hand side is a polynomial in b . \square

Properties of the polynomial $T_n(b)$ will be described in future publications. We now simply observe that its coefficients are *unimodal*. Recall that a polynomial

$$(4.17) \quad P_n(b) = \sum_{k=0}^n c_k b^k$$

is called *unimodal* if there is an index n^* , such that $c_k \leq c_{k+1}$ for $0 \leq k \leq n^*$ and $c_k \geq c_{k+1}$ for $n^* < k \leq n$. That is, the sequence of coefficients of P_n has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [2] and [6] information about this property. We now use the result of [1] to establish the unimodality of T_n .

Theorem 4.6. Suppose $c_k > 0$ is a nondecreasing sequence. Then $P(x+1)$ is unimodal.

Therefore we consider the polynomial $S_n(b) := T_n(b-1)$. It satisfies the recurrence

$$(4.18) \quad S_n(b) = b(n-2)S_{n-1}(b) + (n-3)! \sum_{r=0}^{n-3} b^r.$$

Now write

$$(4.19) \quad S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k,$$

and conclude that $c_{0,n} = (n-3)!$ and

$$(4.20) \quad c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!,$$

from which it follows that

$$(4.21) \quad c_{k+1,n} - c_{k,n} = (n-2)[c_{k,n-1} - c_{k-1,n-1}].$$

We conclude that $c_{k,n}$ is a nondecreasing sequence.

Theorem 4.7. The polynomial $T_n(b)$ is unimodal.

Conclusions. We have given explicit formulas for integrals of the form

$$(4.22) \quad \int_a^b R(x) \ln x \, dx,$$

where R is a rational function with real poles. Future reports will describe the case of higher powers

$$(4.23) \quad \int_a^b R(x) \ln^m x \, dx,$$

as well as the case of complex poles, based on integrals of the form

$$(4.24) \quad C_n(a, r) := \int_0^b \frac{\ln x \, dx}{(x^2 + r^2)^n}.$$

Acknowledgments. The author wishes to thank Luis Medina for a careful reading of the manuscript. The partial support of NSF-DMS 0409968 is also acknowledged.

References

- [1] G. Boros and V. Moll: *A criterion for unimodality*. Elec. Journal of Combinatorics, **6**, 1999, 1-6.
- [2] F. Brenti: *Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry: an update*. Contemporary Mathematics, **178**, 1994, 71-89.
- [3] I. S. Gradshteyn and I. M. Ryzhik: *Table of Integrals, Series, and Products*, 6th edition, 2000. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York.
- [4] L. Lewin: *Dilogarithms and Associated Functions*, 2nd edition, 1981. Elsevier, North Holland, Amsterdam, The Netherlands.
- [5] V. Moll: *The integrals in Gradshteyn and Ryzhik. Part 1: a family of logarithmic integrals*. Scientia, **14**, 2007, 1-6.
- [6] R. Stanley: *Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry*. Graph Theory and its applications: East and West Jinan, 1986. Annals of the New York Academy of Sciences, **576**, 1989, 500-535.

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Received 27 07 2006, revised 06 11 2006

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