

The integrals in Gradshteyn and Ryzhik. Part 3: Combinations of logarithms and exponentials

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ABSTRACT. We present the evaluation of a family of exponential-logarithmic integrals. These have integrands of the form $P(e^{tx}, \ln x)$ where P is a polynomial. The examples presented here appear in sections 4.33, 4.34 and 4.35 in the classical table of integrals by I. Gradshteyn and I. Ryzhik.

1. Introduction

This is the third in a series of papers dealing with the evaluation of definite integrals in the table of Gradshteyn and Ryzhik [2]. We consider here problems of the form

$$(1.1) \quad \int_0^\infty e^{-tx} P(\ln x) dx,$$

where $t > 0$ is a parameter and P is a polynomial. In future work we deal with the finite interval case

$$(1.2) \quad \int_a^b e^{-tx} P(\ln x) dx,$$

where $a, b \in \mathbb{R}^+$ with $a < b$ and $t \in \mathbb{R}$. The classical example

$$(1.3) \quad \int_0^\infty e^{-x} \ln x dx = -\gamma,$$

where γ is Euler's constant is part of this family. The integrals of type (1.1) are linear combinations of

$$(1.4) \quad J_n(t) := \int_0^\infty e^{-tx} (\ln x)^n dx.$$

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The values of these integrals are expressed in terms of the gamma function

$$(1.5) \quad \Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

and its derivatives.

2. The evaluation

In this section we consider the value of $J_n(t)$ defined in (1.4). The change of variables $s = tx$ yields

$$(2.1) \quad J_n(t) = \frac{1}{t} \int_0^\infty e^{-s} (\ln s - \ln t)^n ds.$$

Expanding the power yields J_n as a linear combination of

$$(2.2) \quad I_m := \int_0^\infty e^{-x} (\ln x)^m dx, \quad 0 \leq m \leq n.$$

An analytic expression for these integrals can be obtained directly from the representation of the *gamma function* in (1.5).

Proposition 2.1. For $n \in \mathbb{N}$ we have

$$(2.3) \quad \int_0^\infty (\ln x)^n x^{s-1} e^{-x} dx = \left(\frac{d}{ds} \right)^n \Gamma(s).$$

In particular

$$(2.4) \quad I_n := \int_0^\infty (\ln x)^n e^{-x} dx = \Gamma^{(n)}(1).$$

PROOF. Differentiate (1.5) n -times with respect to the parameter s . □

Example 2.2. Formula 4.331.1 in [2] states that¹

$$(2.5) \quad \int_0^\infty e^{-\mu x} \ln x dx = -\frac{\delta}{\mu}$$

where $\delta = \gamma + \ln \mu$. This value follows directly by the change of variables $s = \mu x$ and the classical special value $\Gamma'(1) = -\gamma$. The reader will find in chapter 9 of [1] details on this constant. In particular, if $\mu = 1$, then $\delta = \gamma$ and we obtain (1.3):

$$(2.6) \quad \int_0^\infty e^{-x} \ln x dx = -\gamma.$$

The change of variables $x = e^{-t}$ yields the form

$$(2.7) \quad \int_{-\infty}^\infty t e^{-t} e^{-e^{-t}} dt = \gamma.$$

¹The table uses C for the Euler constant.

Many of the evaluations are given in terms of the *polygamma function*

$$(2.8) \quad \psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

Properties of ψ are summarized in Chapter 1 of [4]. A simple representation is

$$(2.9) \quad \psi(x) = \lim_{n \rightarrow \infty} \left(\ln n - \sum_{k=0}^n \frac{1}{x+k} \right),$$

from where we conclude that

$$(2.10) \quad \psi(1) = \lim_{n \rightarrow \infty} \left(\ln n - \sum_{k=1}^n \frac{1}{k} \right) = -\gamma,$$

this being the most common definition of the Euler's constant γ . This is precisely the identity $\Gamma'(1) = -\gamma$.

The derivatives of ψ satisfy

$$(2.11) \quad \psi^{(m)}(x) = (-1)^{m+1} m! \zeta(m+1, x),$$

where

$$(2.12) \quad \zeta(z, q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}$$

is the *Hurwitz zeta function*. This function appeared in [3] in the evaluation of some logarithmic integrals.

Example 2.3. Formula 4.335.1 in [2] states that

$$(2.13) \quad \int_0^{\infty} e^{-\mu x} (\ln x)^2 dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + \delta^2 \right],$$

where $\delta = \gamma + \ln \mu$ as before. This can be verified using the procedure described above: the change of variable $s = \mu x$ yields

$$(2.14) \quad \int_0^{\infty} e^{-\mu x} (\ln x)^2 dx = \frac{1}{\mu} (I_2 - 2I_1 \ln \mu + I_0 \ln^2 \mu),$$

where I_n is defined in (2.4). To complete the evaluation we need some special values: $\Gamma(1) = 1$ is elementary, $\Gamma'(1) = \psi(1) = -\gamma$ appeared above and using (2.11) we have

$$(2.15) \quad \psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)} \right)^2.$$

The value

$$(2.16) \quad \psi'(1) = \zeta(2) = \frac{\pi^2}{6},$$

where $\zeta(z) = \zeta(z, 1)$ is the Riemann zeta function, comes directly from (2.11). Thus

$$(2.17) \quad \Gamma''(1) = \zeta(2) + \gamma^2.$$

Let $\mu = 1$ in (2.13) to produce

$$(2.18) \quad \int_0^{\infty} e^{-x} (\ln x)^2 dx = \zeta(2) + \gamma^2.$$

Similar arguments yields formula **4.335.3** in [2]:

$$(2.19) \quad \int_0^{\infty} e^{-\mu x} (\ln x)^3 dx = -\frac{1}{\mu} \left[\delta^3 + \frac{1}{2} \pi^2 \delta - \psi''(1) \right],$$

where, as usual, $\delta = \gamma + \ln \mu$. The special case $\mu = 1$ now yields

$$(2.20) \quad \int_0^{\infty} e^{-x} (\ln x)^3 dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma + \psi''(1).$$

Using the evaluation

$$(2.21) \quad \psi''(1) = -2\zeta(3)$$

produces

$$(2.22) \quad \int_0^{\infty} e^{-x} (\ln x)^3 dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma - 2\zeta(3).$$

Problem 2.4. In [1], page 203, we introduced the notion of *weight* for some real numbers. In particular, we have assigned $\zeta(j)$ the weight j . Differentiation increases the weight by 1, so that $\zeta'(3)$ has weight 4. The task is to check that the integral

$$(2.23) \quad I_n := \int_0^{\infty} e^{-x} (\ln x)^n dx$$

is a homogeneous form of weight n .

3. A small variation

Similar arguments are now employed to produce a larger family of integrals. The representation

$$(3.1) \quad \int_0^{\infty} x^{s-1} e^{-\mu x} dx = \mu^{-s} \Gamma(s),$$

is differentiated n times with respect to the parameter s to produce

$$(3.2) \quad \int_0^{\infty} (\ln x)^n x^{s-1} e^{-\mu x} dx = \left(\frac{d}{ds} \right)^n [\mu^{-s} \Gamma(s)].$$

The special case $n = 1$ yields

$$(3.3) \quad \begin{aligned} \int_0^{\infty} x^{s-1} e^{-\mu x} \ln x dx &= \frac{d}{ds} [\mu^{-s} \Gamma(s)] \\ &= \mu^{-s} (\Gamma'(s) - \ln \mu \Gamma(s)) \\ &= \mu^{-s} \Gamma(s) (\psi(s) - \ln \mu). \end{aligned}$$

This evaluation appears as **4.352.1** in [2]. The special case $\mu = 1$ yields

$$(3.4) \quad \int_0^{\infty} x^{s-1} e^{-x} \ln x dx = \Gamma'(s),$$

that is **4.352.4** in [2].

Special values of the gamma function and its derivatives yield more concrete evaluations. For example, the functional equation

$$(3.5) \quad \psi(x+1) = \psi(x) + \frac{1}{x},$$

that is a direct consequence of $\Gamma(x+1) = x\Gamma(x)$, yields

$$(3.6) \quad \psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}.$$

Replacing $s = n+1$ in (3.3) we obtain

$$(3.7) \quad \int_0^\infty x^n e^{-\mu x} \ln x \, dx = \frac{n!}{\mu^{n+1}} \left(\sum_{k=1}^n \frac{1}{k} - \gamma - \ln \mu \right),$$

that is **4.352.2** in [2].

The final formula of Section **4.352** in [2] is **4.352.3**

$$\int_0^\infty x^{n-1/2} e^{-\mu x} \ln x \, dx = \frac{\sqrt{\pi} (2n-1)!!}{2^n \mu^{n+1/2}} \left[2 \sum_{k=1}^n \frac{1}{2k-1} - \gamma - \ln(4\mu) \right].$$

This can also be obtained from (3.3) by using the classical values

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \frac{\sqrt{\pi}}{2^n} (2n-1)!! \\ \psi\left(n + \frac{1}{2}\right) &= -\gamma + 2 \left(\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right). \end{aligned}$$

The details are left to the reader.

Section **4.353** of [2] contains three peculiar combinations of integrands. The first two of them can be verified by the methods described above: formula **4.353.1** states

$$(3.8) \quad \int_0^\infty (x-\nu)x^{\nu-1} e^{-x} \ln x \, dx = \Gamma(\nu),$$

and **4.353.2** is

$$(3.9) \quad \int_0^\infty (\mu x - n - \frac{1}{2})x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}}.$$

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