THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 22: FRULLANI INTEGRALS. PRELIMINARY VERSION: LAST UPDATE SEPTEMBER 20, 2006.

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Abstract. We present a systematic derivation of some of the definite integrals in the classical table of Gradshteyn and Rhyzik that are of Frullani type.

1. Introduction

The table of integrals [1] contains many evaluations of the form

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = [f(0)-f(\infty)]\,\ln\left(\frac{b}{a}\right). \tag{1.1}$$

Expressions of this type are called *Frullani integrals*. The conditions on f that guarantee the result are ????

2. Some examples

The evaluation of 3.434.2 in [1]:

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\left(\frac{b}{a}\right) \tag{2.1}$$

• 3.434.2 corresponds to the function $f(x)=e^{-x}$. The change of variables $t=e^{-x}$ yields

$$\int_0^1 \frac{t^{a-1}-t^{b-1}}{\ln t}\,dt = \ln\left(\frac{a}{b}\right). \tag{2.2}$$

• 4.267.8 This is 4.267.8 in [1].

If we now take

$$f(x) = \frac{e^{-px}-e^{-qx}}{x}, \tag{2.3}$$

with p, $q>0$, then $f(\infty)=0$ and

$$f(0) = \lim_{x\to 0}\frac{e^{-qx}-e^{-px}}{x} = q-p. \tag{2.4}$$

Then Frullani's theorem yields

$$\int_0^\infty \left(\frac{e^{-aqx}-e^{-apx}}{ax} - \frac{e^{-bqx}-e^{-bpx}}{bx}\right)\frac{dx}{x} = (q-p)\ln\left(\frac{b}{a}\right).$$

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This can be written as

$$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{a} - \frac{e^{-bqx} - e^{-bpx}}{b} \right) \frac{dx}{x^2} = (q-p) \ln \left(\frac{b}{a} \right),$$

and this is the way it appears in 3.436 of [1]. • 3.436

Now choose

$$f(x) = \frac{x}{1-e^{-x}} \exp(-ce^x). \tag{2.5}$$

Then Frullani's theorem yields 3.329 in view of $f(0) = e^{-c}$ and $f(\infty) = 0$: • 3.329

$$\int_0^\infty \left(\frac{a \exp(-ce^{ax})}{1-e^{-ax}} - \frac{b \exp(-ce^{bx})}{1-e^{-bx}} \right) dx = e^{-c} \ln \left(\frac{b}{a} \right).$$

The next example uses

$$f(x) = (x+c)^{-\mu}, \tag{2.6}$$

to produce

$$\int_0^\infty \frac{(ax+c)^{-\mu} - (bx+c)^{-\mu}}{x} dx = c^{-\mu} \ln \left(\frac{b}{a} \right). \tag{2.7}$$

This is 3.232 in [1]. • 3.232

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References

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