

**THE LAPLACE TRANSFORM OF THE DIGAMMA FUNCTION:
AN INTEGRAL DUE TO GLASSER, MANNA AND OLOA**

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ABSTRACT. The definite integral

$$M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)}$$

is related to the Laplace transform of the digamma function

$$L(a) := \int_0^\infty e^{-as} \psi(s+1) ds,$$

by $M(a) = L(a) + \gamma/a$ when $a > \ln 2$. Certain analytic expressions for $M(a)$ in the complementary range, $0 < a \leq \ln 2$, are also provided.

1. INTRODUCTION

The classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik [7] contains a large collection organized in sections according to the form of the integrand. In each section one finds significant variation on the complexity of the integrals. For example, section 4.33 – 4.34, with the title *Combinations of logarithms and exponentials*, presents the elementary formula 4.331.1. For $a > 0$,

$$(1.1) \quad \int_0^\infty e^{-ax} \ln x dx = -\frac{\gamma + \ln a}{a},$$

where γ is the *Euler constant*

$$(1.2) \quad \gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n.$$

Contained in the same section are the more elaborate 4.332.1 and 4.325.6:

$$\int_0^\infty \frac{\ln x dx}{e^x + e^{-x} - 1} = \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{x^2 - x + 1} = \frac{2\pi}{\sqrt{3}} \left(\frac{5}{6} \ln 2\pi - \ln \Gamma \left(\frac{1}{6} \right) \right).$$

The difficulty involved in the evaluation of a definite integral is hard to measure from the complexity of the integrand. For instance, the evaluation of *Vardi's integral*,

$$(1.3) \quad \int_{\pi/4}^{\pi/2} \ln \ln \tan x dx = \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \left(\frac{\Gamma(\frac{3}{4}) \sqrt{2\pi}}{\Gamma(\frac{1}{4})} \right),$$

that appears as 4.229.7 in [7], requires a reasonable amount of Number Theory. The second integral form is 4.325.4, found in the section entitled *Combinations of logarithmic functions of more complicated arguments and powers*. The reader will find in [15] a discussion of this formula.

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It is a remarkable fact that combinations of elementary functions in the integrand often exhibit definite integrals whose evaluation is far from elementary. A systematic study of the formulas in [7] has been initiated in the series [1, 2, 9, 10, 11, 12]. These papers are organized according to the *combinations* appearing in the integrand. Even the elementary cases, such as the combination of logarithms and rational functions discussed in [2], entail interesting results. The evaluation

$$(1.4) \quad \int_0^b \frac{\ln t \, dt}{(1+t)^{n+1}} = \frac{1}{n} [1 - (1+b)^{-n}] \ln b - \frac{1}{n} \ln(1+b) \\ - \frac{1}{n(1+b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!} \binom{n-1}{j} |s(j+1, 2)| b^j,$$

for $b > 0$ and $n \in \mathbb{N}$, produces an explicit formula for the case where the rational function has a single pole. Here, $|s(n, k)|$ are the *unsigned Stirling numbers of the first kind*, which count the number of permutations of n letters having exactly k cycles. The case of a purely imaginary pole

$$\int_0^x \frac{\ln t \, dt}{(1+t^2)^{n+1}} = \frac{\binom{2n}{n}}{2^{2n}} \left[g_0(x) + p_n(x) \ln x - \sum_{k=0}^{n-1} \frac{\tan^{-1} x + p_k(x)}{2k+1} \right],$$

is expressed in terms of the rational function

$$(1.5) \quad p_n(x) = \sum_{j=1}^n \frac{2^{2j}}{2^j \binom{2j}{j}} \frac{x}{(1+x^2)^j},$$

and with

$$(1.6) \quad g_0(x) = \ln x \tan^{-1} x - \int_0^x \frac{\tan^{-1} t}{t} dt.$$

The special case $x = 1$ becomes

$$(1.7) \quad \int_0^1 \frac{\ln t \, dt}{(1+t^2)^{n+1}} = -2^{-2n} \binom{2n}{n} \left(G + \sum_{k=0}^{n-1} \frac{\frac{\pi}{4} + p_k(1)}{2k+1} \right),$$

where

$$(1.8) \quad G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is *Catalan's constant*. The values

$$(1.9) \quad p_k(1) = \sum_{j=1}^k \frac{2^j}{2^j \binom{2j}{j}}$$

do not admit a closed-form (in the sense of [14]), but they do satisfy the three-term recurrence

$$(1.10) \quad (2k+1)p_{k+1}(1) - (3k+1)p_k(1) + kp_{k-1}(1) = 0.$$

The study of definite integrals where the integrand is a combination of powers, logarithms and trigonometric functions was initiated by Euler [5], with the evaluation of

$$(1.11) \quad \int_0^{\pi/2} x \ln(2 \cos x) dx = -\frac{7}{16} \zeta(3),$$

and

$$(1.12) \quad \int_0^{\pi/2} x^2 \ln(2 \cos x) dx = -\frac{\pi}{4} \zeta(3),$$

which appear in his study of the *Riemann zeta* function at the odd integers. These type of integrals have been investigated in [8], [16]. The *intriguing integral* of D. and J. Borwein [3],

$$(1.13) \quad \int_0^{\pi/2} x^2 \ln^2(2 \cos x) dx = \frac{11\pi}{16} \zeta(4) = \frac{11\pi^5}{1440},$$

was first conjectured on the basis of a numerical computation by Enrico Au-Yueng while an undergraduate student at the University of Waterloo. A nice example of Experimental Mathematics in action.

Recently O. Oloa considered the integral

$$(1.14) \quad M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)},$$

and the special value

$$(1.15) \quad M(0) = \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2 \cos x)} = \frac{1}{2}(1 + \ln(2\pi) - \gamma)$$

is established in [13].

Oloa's method of proof relies on the expansion

$$(1.16) \quad \frac{x^2}{x^2 + \ln^2(2 \cos x)} = x \sin 2x + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{a_n}{n!} - \frac{a_{n+1}}{(n+1)!} \right) x \sin(2nx).$$

Here

$$(1.17) \quad a_n := \int_0^1 (t)_n dt,$$

where $(t)_n = t(t+1) \cdots (t+n-1)$ the *Pochhammer symbol*. The standard relation

$$(1.18) \quad (t)_n = \sum_{k=1}^n |s(n, k)| t^k$$

gives

$$(1.19) \quad a_n = \sum_{k=1}^n \frac{|s(n, k)|}{k+1}.$$

M. L. Glasser and D. Manna [6] introduced the function

$$(1.20) \quad L(a) := \int_0^{\infty} e^{-as} \psi(s+1) ds,$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the *digamma function*. After integrating by parts and making us of (1.1), one finds

$$(1.21) \quad L(a) = -\gamma - \ln a + a \int_0^{\infty} e^{-at} \ln \Gamma(t) dt.$$

The main result in [6] gives a relation between $M(a)$ and $L(a)$.

Theorem 1.1. *If $a > \ln 2$, then*

$$M(a) = L(a) + \frac{\gamma}{a}.$$

That is, for $a > \ln 2$,

$$(1.22) \quad M(a) = \frac{\gamma}{a} - \gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t) dt.$$

The proof in [6] begins with the representation

$$(1.23) \quad \int_0^{\pi/2} \cos^\nu x \cos ax dx = \frac{\pi \Gamma(\nu + 2)}{2^{\nu+1} (\nu + 1) \Gamma(1 + \frac{\nu}{2} + \frac{a}{2}) \Gamma(1 + \frac{\nu}{2} - \frac{a}{2})},$$

(3.621.9 in [7]). Differentiating with respect to a , evaluating at $a = s$, and using $\psi(1) = -\gamma$ yields

$$(1.24) \quad \psi(s + 1) = \frac{2^{s+2}}{\pi} \int_0^{\pi/2} x \cos^s x \sin(sx) dx - \gamma.$$

Replacing (1.24) into (1.20) produces

$$(1.25) \quad L(a) + \frac{\gamma}{a} = -\frac{4}{\pi} \operatorname{Im} \int_0^\infty \int_0^{\pi/2} x e^{s(\ln[2e^{-a} \cos x] - ix)} dx ds.$$

The identity (1.22) follows from evaluating the s -integral as

$$(1.26) \quad \int_0^\infty e^{s(\ln[2e^{-a} \cos x] - ix)} ds = \frac{1}{ix - \ln[2e^{-a} \cos x]}.$$

The authors of [6] produced a series expansion for $M(a)$, which they recognize as a hypergeometric function in two variables, and state that *this strongly suggests that for general value of a , no further progress is possible*. The hypergeometric expression gives

$$(1.27) \quad M(0) = 1 + \frac{1}{2} \int_0^1 t(1-t) {}_3F_2(1, 1, 2-t; 2, 3; 1) dt,$$

on which they invoke

$$(1.28) \quad {}_3F_2(1, 1, 2-t; 2, 3; 1) = \frac{2(1-\gamma-\psi(t+1))}{1-t}$$

to give a new proof of (1.15).

The graph of $M(a)$ shown in Figure 1, obtained by the numerical integration of (1.14), has a well-defined *cusp* at $a = \ln 2$. In this paper, analytic expressions for both branches of $M(a)$ are provided. The region $a > \ln 2$, determined in [6], has been reviewed in the present section. The corresponding expressions for $0 < a < \ln 2$ is the content of the next section.

2. THE CASE $0 < a < \ln 2$

The representation

$$(2.1) \quad M(a) = -\frac{e^a}{2\pi} \operatorname{Im} \int_0^1 e^{-at} \int_{-\pi}^\pi \frac{x(1+e^{ix})^t}{1-e^a+e^{ix}} dx dt,$$

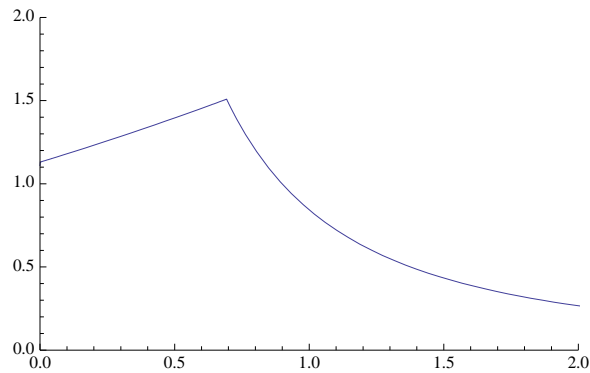


FIGURE 1. The graph of $M(a)$ for $0 \leq a \leq 2$

was established in [6]. Their proof is replicated here for sake of the reader's convenience. The identity

$$(2.2) \quad \operatorname{Im} \frac{x}{ix + \ln [2e^{-a} \cos x]} = \frac{x^2}{x^2 + \ln^2 [2e^{-a} \cos x]}$$

yields

$$(2.3) \quad M(a) = \frac{4}{\pi} \operatorname{Im} \int_0^{\pi/2} \frac{x dx}{ix + \ln [2e^{-a} \cos x]}.$$

If $a > \ln 2$, then

$$(2.4) \quad \int_0^\infty e^{s \ln [2e^{-a} \cos x] + ix} ds = \frac{1}{ix + \ln [2e^{-a} \cos x]}.$$

This implies

$$(2.5) \quad M(a) = \frac{2}{\pi} \operatorname{Im} \int_{-\pi/2}^{\pi/2} \int_0^\infty x e^{s(\ln [2e^{-a} \cos x] + ix)} dx ds,$$

where one uses the fact that the imaginary part of the integrand is an *even* function of x . One more identity,

$$(2.6) \quad e^{isx} \cdot e^{s \ln [2e^{-a} \cos x]} = e^{s \ln [e^{-a}(1+e^{2ix})]},$$

and the change of variables $x \mapsto x/2$, give the equality

$$(2.7) \quad M(a) = \frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \int_0^\infty x e^{s \ln [e^{-a}(1+e^{ix})]} ds dx.$$

Evaluating the s -integral yields

$$(2.8) \quad M(a) = -\frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \frac{x dx}{\ln [e^{-a}(1+e^{ix})]}.$$

The formula

$$(2.9) \quad \frac{1}{\ln u} = \int_0^1 \frac{u^t dt}{u-1}$$

now gives (2.1) from (2.8).

Note 2.1. *The proof outlined above is valid for $a > \ln 2$, but (2.1) holds for $a > 0$.*

Notation. Define $b := e^a - 1$ and let $0 < a < \ln 2$ so that $0 < b < 1$.

The terms $(1 + e^{ix})^t$ and $1/(1 - be^{-ix})$ from (2.1) are now expanded in power series to produce

$$M(a) = -\frac{e^a}{2\pi} \int_0^1 \int_{-\pi}^{\pi} x e^{-at} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b^j \binom{t}{k} \sin[x(k-j-1)] dx dt.$$

The term corresponding to $k = j+1$ disappears and a computation of the x -integral gives

$$(2.10) \quad \begin{aligned} M(a) &= e^a \int_0^1 e^{-at} \sum_{j=0}^{\infty} \sum_{k=0}^j (-1)^{j-k} \frac{b^j \binom{t}{k}}{j+1-k} dt \\ &+ e^a \int_0^1 e^{-at} \sum_{j=1}^{\infty} b^j \sum_{\nu=1}^j \frac{(-1)^\nu}{\nu} \binom{t}{\nu+j} dt. \end{aligned}$$

Lemma 2.1. *Let $t \in \mathbb{R}$ and $j \in \mathbb{N} \cup \{0\}$. Then*

$$(2.11) \quad \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu} \binom{t}{\nu+j} = \binom{t}{j} [\psi(j+1) - \psi(t+1)].$$

Proof. The integral representation (3.268.2 in [7])

$$(2.12) \quad \psi(p+1) - \psi(q+1) = -\int_0^1 \frac{x^p - x^q}{1-x} dx$$

yields

$$(2.13) \quad \psi(p+1) - \psi(q+1) = \sum_{j=1}^{\infty} (-1)^{j-1} \left(\binom{p}{j} - \binom{q}{j} \right).$$

The result now follows from the identity

$$(2.14) \quad \binom{t}{k}^{-1} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \binom{t}{m+k} - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \binom{t}{m} = \sum_{m=1}^k \frac{(-1)^{m-1}}{m} \binom{k}{m}.$$

Apply the difference operator $\Delta a(k) := a(k+1) - a(k)$ and use

$$\binom{t}{k+1}^{-1} \binom{t}{m+k+1} - \binom{t}{k}^{-1} \binom{t}{m+k} = -\frac{m}{k+1} \binom{t}{k+1}^{-1} \binom{t+1}{m+k+1}$$

to write the derived equation as

$$(2.15) \quad -\frac{\binom{t}{k+1}^{-1}}{k+1} \sum_{m=1}^{\infty} (-1)^m \binom{t+1}{m+k+1} = \Delta \sum_{m=1}^k \frac{(-1)^m}{m} \binom{k}{m}.$$

The left hand side of (2.14) reduces to $-1/(k+1)$ in view of the classical identity

$$(2.16) \quad \sum_{m=1}^{\infty} (-1)^{m-1} \binom{t+1}{m+k+1} = \binom{t}{k+1}.$$

A simple evaluation of the right hand side in (2.14) also produces $-1/(k+1)$. Therefore both sides of (2.14) are, up to a constant, the *harmonic number* H_k . The special case $k = 0$ shows that this constant vanishes. \square

Continuing from (2.10), it follows that

$$(2.17) \quad \begin{aligned} M(a) &= e^a \int_0^1 \sum_{j=0}^{\infty} b^j \sum_{k=0}^j \frac{(-1)^{j-k} \binom{t}{k}}{j+1-k} + \frac{e^a}{b} \int_0^1 e^{-at} \sum_{j=1}^{\infty} b^j \binom{t}{j} \psi(j+1) dt \\ &\quad - \frac{e^a}{b} \int_0^1 e^{-at} \sum_{j=1}^{\infty} b^j \binom{t}{j} \psi(t+1) dt \equiv M_1 + M_2 + M_3. \end{aligned}$$

To simplify M_1 , observe

$$\begin{aligned} \sum_{j=0}^{\infty} b^j \sum_{k=0}^j \frac{(-1)^{j-k} \binom{t}{k}}{j+1-k} &= \sum_{k=0}^{\infty} b^k \binom{t}{k} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} b^{\nu}}{\nu+1} \\ &= \frac{\ln(1+b)}{b} \sum_{k=0}^{\infty} \binom{t}{k} b^k = \frac{ae^{at}}{b}. \end{aligned}$$

Thus, $M_1 = a/(1 - e^{-a})$.

The reduction of M_2 employs the following result.

Lemma 2.2. *If $0 < a < \ln 2$, then*

$$(2.18) \quad \int_0^1 e^{-at} \sum_{j=0}^{\infty} b^j \binom{t}{j} \psi(j+1) dt = \ln(1 - e^{-a}) + \int_1^{\infty} \frac{e^{-at}}{t} dt.$$

Proof. The Stirling numbers $s(j, k)$ satisfy

$$(2.19) \quad j! \binom{t}{j} = \sum_{k=0}^j s(j, k) t^k,$$

so that

$$(2.20) \quad \int_0^1 e^{-at} \sum_{j=0}^{\infty} b^j \binom{t}{j} \psi(j+1) dt = \frac{e^{-a} b^{\gamma}}{a} + e^{-a} \sum_{j=1}^{\infty} (b^{j+1} \alpha_j - b^j \alpha_{j-1}) \psi(j+1),$$

with

$$(2.21) \quad \alpha_j(a) = \frac{1}{j!} \sum_{k=0}^j \frac{s(j, k) k!}{a^{k+1}}.$$

The result now follows from integration by parts and the identity

$$(2.22) \quad \sum_{j=k}^{\infty} \frac{s(j, k) b^j}{j!} = \frac{\ln^k(1+b)}{k!}.$$

□

Therefore,

$$(2.23) \quad M_2 = \frac{\ln(1 - e^{-a})}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_1^{\infty} \frac{e^{-at}}{t} dt.$$

Finally,

$$M_3 = -\frac{e^a}{b} \int_0^1 e^{-at} \left(\sum_{j=1}^{\infty} \binom{t}{j} b^j \right) \psi(t+1) dt = -\frac{e^a}{b} \int_0^1 (1 - e^{-at}) \psi(t+1) dt.$$

A direct computation shows that $\int_0^1 \psi(t+1) dt = 0$, and integration by parts gives

$$(2.24) \quad M_3 = \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t+1) dt.$$

The identity $\ln \Gamma(t+1) = \ln \Gamma(t) + \ln t$ now yields

$$M_3 = \frac{a}{1 - e^{-a}} \left(\int_0^1 e^{-at} \ln t dt + \int_0^1 e^{-at} \ln \Gamma(t) dt \right).$$

Replacing (2.17), (2.19) and (2.21) into (2.17) provides the an expression for $M(a)$:

$$(2.25) \quad M(a) = \frac{a}{1 - e^{-a}} + \frac{\gamma}{a} + \frac{\ln(1 - e^{-a})}{1 - e^{-a}} \\ + \frac{a}{1 - e^{-a}} \int_0^{\infty} e^{-at} \ln t dt + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt.$$

The term γ/a comes from the index $j = 0$ in the sum (2.18). The main result presented here now follows from (1.1). This settles a conjecture of O. Oloa presented in [13].

Theorem 2.1. *If $0 < a < \ln 2$, then*

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) - \gamma - \ln a}{1 - e^{-a}} + \frac{a}{1 - e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) dt.$$

The above result is complementary to Theorem 1.1.

Corollary 2.1. *If $0 < a < \ln 2$, then*

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) + \Gamma(0, a)}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_0^1 e^{-at} \psi(t+1) dt;$$

where $\Gamma(0, a)$ is the incomplete gamma function.

Proof. Split up the first integral in (2.25) and integrate by parts. □

The derivative of (2.1) at $a = 0$, the classical values

$$(2.26) \quad \int_0^1 \ln \Gamma(t) dt = \frac{1}{2} \ln 2\pi$$

and

$$(2.27) \quad \int_0^1 t \ln \Gamma(t) dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6} \ln 2\pi - \frac{\gamma}{12},$$

obtained in [4], give

$$(2.28) \quad \int_0^{\pi/2} \frac{x^2 \ln(2 \cos x) dx}{(x^2 + \ln^2(2 \cos x))^2} = \frac{7\pi}{192} + \frac{\pi \ln 2\pi}{96} - \frac{\zeta'(2)}{16\pi}.$$

Further differentiation of (2.1) produces the evaluation of a family of integrals similar to (2.28).

The integral in (2.1) can be expressed in an alternative form. Define

$$(2.29) \quad \Lambda(z) := \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{j}{j^2 + z^2} - \ln n \right).$$

Observe that $\Lambda(0) = \gamma$, so $\Lambda(z)$ is a generalization of Euler's constant.

Lemma 2.3. *Let $a > 0$, $c = 1 - e^{-a}$ and define $A := \ln 2\pi + \gamma$. Then*

$$(2.30) \quad \int_0^1 e^{-at} \ln \Gamma(t) dt = \frac{A(a-c)}{a^2} - \frac{c}{2a} \Lambda\left(\frac{a}{2\pi}\right) + 2c \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}.$$

Proof. Expand the exponential into a MacLaurin series and use the value

$$\begin{aligned} \int_0^1 t^n \ln \Gamma(t) dt &= \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n+1}{2k-1} \frac{(2k)!}{k(2\pi)^{2k}} [A\zeta(2k) - \zeta'(2k)] \\ &\quad - \frac{1}{n+1} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n+1}{2k} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) + \frac{\ln \sqrt{2\pi}}{n+1} \end{aligned}$$

given as (6.14) in [4]. Then interchange the resulting double sums. \square

The next corollary follows from the identity $M(a) = L(a) + \gamma/a$.

Corollary 2.2. *If $0 < a < \ln 2$ and $c = 1 - e^{-a}$, then*

$$(2.31) \quad \int_0^{\infty} e^{-at} \ln \Gamma(t) dt = -\frac{\gamma + \ln a}{ace^a} + \frac{A(a-c)}{a^2 c} - \frac{1}{2a} \Lambda\left(\frac{a}{2\pi}\right) + 2 \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}.$$

Lemma 2.4. *Let $f(t) = 2^{-t} \ln \Gamma(t)$. Then*

$$(2.32) \quad \int_0^{\infty} f(t) dt = 2 \int_0^1 f(t) dt - \frac{\gamma + \ln \ln 2}{\ln 2}$$

$$(2.33) \quad \int_0^{\infty} t f(t) dt = 2 \int_0^1 (t+1) f(t) dt - \frac{(\gamma + \ln \ln 2)(1 + 2 \ln 2) - 1}{\ln^2 2}.$$

Proof. The function $f(t)$ satisfies $f(t+1) = \frac{1}{2}f(t) + \frac{1}{2}2^{-t} \ln t$. Splitting the integral

$$(2.34) \quad \int_0^{\infty} f(t) dt = \int_0^1 f(t) dt + \int_0^{\infty} f(t+1) dt$$

and using (1.1) gives the first result. The proof of (2.33) is similar; it only requires differentiating (1.1) with respect to the parameter a . \square

The reader will check that (2.32) is equivalent to the continuity of $M(a)$ at $a = \ln 2$. The identity (2.33) provides a proof of the next theorem, which in itself is worthy of singular (pun intended) interest.

Theorem 2.2. *The jump of $M'(a)$ at $a = \ln 2$ is 4.*

3. CONCLUSIONS

The integral

$$M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a} \cos x)}$$

satisfies

$$(3.1) \quad M(a) = \frac{\gamma}{a} + \int_0^\infty e^{-at} \psi(t+1) dt$$

for $a > \ln 2$ and

$$M(a) = \frac{\gamma}{a} + \frac{a + \ln(1 - e^{-a}) + \Gamma(0, a)}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_0^1 e^{-at} \psi(t+1) dt$$

for $0 < a \leq \ln 2$.

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