THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 10: EVALUATION BY SERIES.
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Abstract. The table of Gradshteyn and Rhyzik contains many integrals that can be evaluated by expanding the integrand in series. Some examples are discussed.

1. Introduction

The table of integrals [1] contains a large variety of definite integrals that can be evaluated by expanding the integrand. The idea is remarkably simple: to evaluate

$$I = \int_a^b f(x)\,dx \tag{1.1}$$

one chooses a set of functions $\{f_n : n \in \mathbb{N}\}$ for which it is possible to expand

$$f(x) = \sum_{n=1}^{\infty} a_n f_n, \tag{1.2}$$

uniformly on $[a,b]$. Then, with

$$b_n = \int_a^b f_n(x)\,dx \tag{1.3}$$

we obtain

$$I = \sum_{n=1}^{\infty} a_n b_n. \tag{1.4}$$

In order to obtain a simpler form of the integral I, it is required to identify the series in (1.4).

2. Some hypergeometric examples

The *hypergeometric function* is defined by

$${}_pF_q\left(a_1, \cdots a_p; b_1, \cdots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}. \tag{2.1}$$

We have used the notation

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1) \tag{2.2}$$

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for the *Pochhammer symbol*.

The first example in this section is 3.311.4 in [1]: • 3.311.4

$$\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \sum_{k=0}^\infty \frac{a^k}{q+kp}. \tag{2.3}$$

Expanding the integrand as a geometric series, we have

$$\frac{1}{1-ae^{-px}} = \sum_{k=0}^\infty a^k e^{-kpx}, \tag{2.4}$$

and integrating over $[0,\infty)$ we obtain

$$I = \sum_{k=0}^\infty a^k \int_0^\infty e^{-(q+kp)x}\,dx = \sum_{k=0}^\infty \frac{a^k}{q+kp}. \tag{2.5}$$

It is possible to identify the sum as a hypergeometric function:

$$\sum_{k=0}^\infty \frac{a^k}{q+kp} = \frac{1}{p}\sum_{k=0}^\infty \frac{a^k}{k+c}, \tag{2.6}$$

where $c = q/p$. Now use $k! = (1)_k$ and

$$k+c = \frac{c\,(c+1)_k}{(c)_k} \tag{2.7}$$

to write

$$\sum_{k=0}^\infty \frac{a^k}{q+kp} = \frac{1}{q}\sum_{k=0}^\infty \frac{(c)_k\,(1)_k}{(1+c)_k}\,\frac{a^k}{k!}. \tag{2.8}$$

We conclude that

$$\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \frac{1}{q}\,{}_2F_1\left(\frac{q}{p},\, 1;\, 1+\frac{q}{p};\, a\right). \tag{2.9}$$

3. A PRODUCT OF LOGARITHMS

The value of 4.221.1: • 4.221.1

$$\int_0^1 \ln x\,\ln(1-x)\,dx = 2-\frac{\pi^2}{6} \tag{3.1}$$

can be obtained from the expansion

$$\ln(1-x) = -\sum_{k=1}^\infty \frac{x^k}{k}. \tag{3.2}$$

We obtain

$$I = -\sum_{k=1}^\infty \frac{1}{k}\int_0^1 x^k\,\ln x\,dx \tag{3.3}$$

and the integral can be evaluated by integration by parts to produce

$$\int_0^1 x^k\,\ln x\,dx = -\frac{1}{(k+1)^2}. \tag{3.4}$$

Therefore

$$I = \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} \tag{3.5}$$

and using

$$\frac{1}{k(k+1)^2} = \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \tag{3.6}$$

we obtain the result.

• 4.221.2 The evaluation of 4.221.2:

$$\int_0^1 \ln x \, \ln(1+x)\, dx = 2 - \frac{\pi^2}{12} - 2\ln 2 \tag{3.7}$$

can be obtained by using the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \tag{3.8}$$

and replacing in the integral we obtain

$$I = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 x^k \, \ln x \, dx. \tag{3.9}$$

Integration by parts produces

$$\int_0^1 x^k \, \ln x \, dx = -\frac{1}{(k+1)^2}, \tag{3.10}$$

so we have

$$I = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)^2}. \tag{3.11}$$

Expanding

$$\frac{1}{k(k+1)^2} = \frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \tag{3.12}$$

and using the values

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\ln 2 \text{ and } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}, \tag{3.13}$$

we obtain the result.

• 4.221.3 The evaluation of 4.221.3:

$$\int_0^1 \ln\left(\frac{1-ae^{-t}}{1-a}\right) \frac{dx}{\ln x} = -\sum_{k=1}^{\infty} \frac{a^k}{k} \, \ln(1+k) \tag{3.14}$$

is obtained by using the change of variables $x = e^{-t}$ so that

$$I = -\int_0^{\infty} \ln\left(\frac{1-ae^{-t}}{1-a}\right) \frac{e^{-t}}{t} \, dt. \tag{3.15}$$

The expansions

$$\ln(1-ae^{-t}) = -\sum_{k=1}^{\infty} \frac{a^k}{k} e^{-kt} \text{ and } \ln(1-a) = -\sum_{k=1}^{\infty} \frac{a^k}{k} \tag{3.16}$$

produce

$$I = \sum_{k=1}^{\infty} \frac{a^k}{k} \int_0^{\infty} \frac{e^{-kt}-1}{t} \, e^{-t} dt. \tag{3.17}$$

The integral

$$g(k) = \int_0^{\infty} \frac{e^{-kt}-1}{t} e^{-t} \, dt \tag{3.18}$$

satisfies $g(0) = 0$ and $g'(k) = -1/(k+1)$, thus $g(k) = -\ln(1+k)$ as required.

The series in the answer can be expressed in terms of the *polylogarithm function*

$$\text{Li}_b(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^b}. \tag{3.19}$$

Indeed,

$$f(a) := \sum_{k=1}^{\infty} \frac{a^k \, \ln(1+k)}{k} \tag{3.20}$$

satisfies

$$f'(a) = \sum_{k=1}^{\infty} a^{k-1} \ln(1+k) = \frac{1}{a^2} \sum_{k=2} a^k \ln k. \tag{3.21}$$

This final series is now identified as

$$\sum_{k=2} a^k \ln k = \frac{\partial}{\partial b} \text{Li}_{-b}(a) \Big|_{b=0}. \tag{3.22}$$

4. An integral involving the binomial theorem

The evaluation 3.194.8: • 3.194.8

$$\int_0^1 \frac{x^{n-1} \, dx}{(1+x)^m} = 2^{-n} \sum_{k=0}^{\infty} \binom{m-n-1}{k} \frac{(-2)^{-k}}{n+k} \tag{4.1}$$

can be obtained by using the binomial theorem. Indeed, the change of variables $t = x/(1+x)$ produces

$$I = \int_0^{1/2} \frac{t^{n-1} \, dt}{(1-t)^{n-m+1}} \tag{4.2}$$

and using the expansion

$$(1-t)^{m-n-1} = \sum_{k=0}^{\infty} \binom{m-n-1}{k} (-t)^k \tag{4.3}$$

in (4.2) we obtain the stated result.

5. An integral involving polylogarithms

The evaluation of 3.411.5:

$$\int_0^{\log 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12} \tag{5.1}$$

• 3.411.5

can be obtained by expanding the integrand as

$$\frac{x}{1-e^{-x}} = \sum_{k=0}^{\infty} xe^{-kx} \tag{5.2}$$

and then integration by parts shows that

$$\int_0^{\log 2} xe^{-kx}\,dx = -\log 2 \frac{2^{-k}}{k} + \frac{1-2^{-k}}{k^2}. \tag{5.3}$$

We conclude that

$$\int_0^{\log 2} \frac{x\,dx}{1-e^{-x}} = \frac{\log^2 2}{2} - \log 2 \sum_{k=1}^{\infty} \frac{2^{-k}}{k} + \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{2^{-k}}{k^2},$$

where we have separated the term for $k=0$.

From the expansion

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \tag{5.4}$$

we obtain

$$\sum_{k=1}^{\infty} \frac{2^{-k}}{k} = -\log\frac{1}{2} = \log 2. \tag{5.5}$$

The last series is identified as PolyLog$[2, \tfrac{1}{2}]$, where

$$\text{PolyLog}[m,x] := \sum_{k=1}^{\infty} \frac{x^k}{k^m}. \tag{5.6}$$

To finish the evaluation we establish the special value:

Proposition 5.1. The polylogarithm function satisfies

$$\text{PolyLog}[2, \tfrac{1}{2}] = \frac{1}{12}\left(\pi^2 - 6\log^2 2\right). \tag{5.7}$$

Proof. Start □

6. Combinations of logarithms and rational functions

The table [1] contains many integrals that are combinations of $\ln x$ and a simple rational function. There are many ways to evaluate these integrals. In this section we illustrate the method of expansion in series.

We begin with 4.231.2:

$$\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}. \tag{6.1}$$

• 4.231.2 Expanding the integrand in a geometric series, we obtain

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \sum_{k=0}^{\infty} \int_0^1 x^k \ln x\,dx. \tag{6.2}$$

The change of variables $x = e^{-t}$ produces

$$\int_0^1 \frac{\ln x}{1-x}\,dx = -\sum_{k=0}^{\infty} \int_0^{\infty} te^{-(k+1)t}\,dt \tag{6.3}$$

and with $u = (k+1)t$ we have

$$\int_0^1 \frac{\ln x}{1-x}\,dx = -\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \int_0^{\infty} ue^{-u}\,du = -\frac{\pi^2}{6},$$

where we have used the value

$$\int_0^{\infty} ue^{-u}\,du = 1, \tag{6.4}$$

and the classical evaluation

$$\sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{6.5}$$

The same argument produces 4.231.1: • 4.231.1

$$\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}. \tag{6.6}$$

Expanding the integrand as before, we get the value of this integral using

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12} \tag{6.7}$$

instead of (6.5).

The identities

$$\frac{x}{1-x} = \frac{1}{1-x} - 1 \text{ and } \frac{1-x}{1+x} = \frac{2}{1+x} - 1 \tag{6.8}$$

produce the values of 4.231.3: • 4.231.3

$$\int_0^1 \frac{x\,\ln x}{1-x}\,dx = 1 - \frac{\pi^2}{6}, \tag{6.9}$$

and produce the values of 4.231.4: • 4.231.4

$$\int_0^1 \frac{1-x}{1+x}\,\ln x\,dx = 1 - \frac{\pi^2}{6}. \tag{6.10}$$

On the other hand, the values of 4.231.14:

$$\int_0^1 \frac{x\,\ln x}{1+x^2}\,dx = -\frac{\pi^2}{48}, \tag{6.11}$$

and 4.231.15: • 4.231.14

$$\int_0^1 \frac{x\,\ln x}{1-x^2}\,dx = -\frac{\pi^2}{24}, \tag{6.12}$$

can be obtained by the change of variables $t = x^2$. • 4.231.15

The partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x} \right) \tag{6.13}$$

• 4.231.13 produces the evaluation of 4.231.13:

$$\int_0^1 \frac{\ln x \, dx}{1-x^2} = -\frac{\pi^2}{8}. \tag{6.14}$$

7. Some integrals involving the exponential function

• 3.342 The evaluation of 3.342:

$$\int_0^1 \exp(-px \, \ln x) \, dx = \int_0^1 x^{-px} \, dx = \frac{1}{p} \sum_{k=1}^{\infty} \left(\frac{p}{k} \right)^k \tag{7.1}$$

can be established by expanding the integrand in series. Indeed,

$$\int_0^1 \exp(-px \, \ln x) \, dx \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k p^k}{k!} \int_0^1 x^k \ln^k dx. \tag{7.2}$$

The change of variables $x = e^{-t}$ gives

$$\begin{aligned} \int_0^1 x^k \ln^k x \, dx &= (-1)^k \int_0^{\infty} t^k e^{-(k+1)t} \, dt \\ &= \frac{(-1)^k}{(k+1)^{k+1}} \int_0^{\infty} s^k e^{-s} \, ds \\ &= \frac{(-1)^k \, k!}{(k+1)^{k+1}}. \end{aligned}$$

Replacing in (7.2) gives the result.

• 3.466.3 Similarly entry 3.466.3:

$$\int_0^1 \frac{e^{x^2}-1}{x^2} \, dx = \sum_{k=1}^{\infty} \frac{1}{k! \, (2k-1)} \tag{7.3}$$

is established by expanding the exponential in the integrand and integrating term by term. We now identify the series as

$$\sum_{k=1}^{\infty} \frac{1}{k! \, (2k-1)} = 1 - e + \sqrt{\pi} \, \text{Erfi}(1), \tag{7.4}$$

where Erfi is the ????????

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