SOLUTION TO PROBLEM #11299 PROPOSED BY PABLO FERNANDEZ REFOLIO

Problem: Show that

$$\prod_{n=2}^{\infty} \frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1} = \frac{e\sqrt{e}}{2\pi}. \tag{1}$$

Solution by T. Amdeberhan and V. Moll, Tulane University, New Orleans, Louisiana.

Consider the partial products

$$P_N := \prod_{n=2}^{N} \frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1} = e^{1-N} \prod_{n=2}^{N} \frac{n^2-1}{n^2} \prod_{n=2}^{N} \left(\frac{n^2}{n^2-1}\right)^{n^2}.$$

The first factor telescopes to

$$\prod_{n=2}^{N} \frac{n^2-1}{n^2} = \prod_{n=2}^{N} \frac{n-1}{n} \times \prod_{n=2}^{N} \frac{n}{n+1} = \frac{N+1}{2N},$$

and the second one can be expressed as

$$\prod_{n=2}^{N} \left(\frac{n^2}{(n-1)(n+1)}\right)^{n^2} = \prod_{n=2}^{N} n^{2n^2} \times \prod_{n=1}^{N-1} n^{-(n+1)^2} \times \prod_{n=3}^{N+1} n^{-(n-1)^2}.$$

Combining these three products, we see that each index n in the range $3 \leq n \leq N-1$ appears with total exponent -2. Therefore we obtain

$$\prod_{n=2}^{N} \left(\frac{n^2}{n^2-1}\right)^{n^2} = \frac{2N^{N^2+2N-1}}{(N-1)!^2(N+1)^{N^2}}.$$

Considering the exceptions $n = 1,\ 2,\ N$ and $N+1$ show that the partial product P_N is given by

$$P_N = e^{1-N} \frac{N^{N^2+2N}}{N!^2\,(N+1)^{N^2-1}}.$$

At this point we invoke Stirling's approximation: $N! \sim N^N e^{-N}\sqrt{2\pi N}$. Thus, for N large, we obtain

$$\prod_{n=2}^{N} \frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1} \sim \frac{e}{2\pi}\,\frac{N+1}{N}\,\frac{e^N}{(1+N^{-1})^{N^2}}. \tag{2}$$

The limit of the last piece is better handled by taking logarithms. By L'Hopital's rule we have

$$\lim_{N\to\infty} \ln\left(\frac{e^N}{(1+N^{-1})^{N^2}}\right) = \lim_{N\to\infty} N - N^2\ln(1+1/N) = \lim_{x\to 0^+} \frac{x-\ln(1+x)}{x^2} = \frac{1}{2},$$

so that $e^N/(1+N^{-1})^{N^2} \to \sqrt{e}$. Replacing in (2) confirms (1).