Lecture note 13. Parabolic differential equations and diffusion processes.

We have several times given this formulation: if u satisfies such and such equation, then the value $u(t_0, x_0)$ of this function at the point (t_0, x_0) is equal to the expectation of a certain functional of X_t, *where* X_t *is the diffusion process with the coefficients* $b(t, x)$, $\sigma(t, x)$ *starting at the time* t_0 *from the point* x_0. Repeating this, or at least keeping this in mind and not forgetting it, infinitely many times can be tiring; so let us introduce a short notation.

The diffusion process with given coefficients $b(t, x)$, $\sigma(t, x)$ starting at time t_0 from a point x_0 will be denoted as $X_t^{t_0, x_0}$.

So the statement including formulas (12.26), (12.27) can be rewritten as follows: Let a function $u(t, x)$, $t_0 \le t \le T$, be once continuously differentiable in t and twice in x, and let it be a solution of the final-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = -L_t u(t, x) + g(t, x), & t_0 \le t \le T, \ x \in (-\infty, \infty), \\ u(T, x) = \varphi(x), & x \in (-\infty, \infty), \end{cases} \tag{13.1}$$

where the operator L_t is applied to twice continuously differentiable functions $f(x)$, $x \in \mathbb{R}$, according to the formula

$$L_t f(x) = \frac{a(t, x)}{2} f''(x) + b(t, x) f'(x). \tag{13.2}$$

Let X_t be a diffusion process being a solution of the stochastic differential equation

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \tag{13.3}$$

where $\sigma(t, x)^2 = a(t, x)$. Let

$$E\big(u(T, X_T^{t_0, x_0})\big), \qquad \int_{t_0}^{T} E\Big(\big(\frac{\partial u}{\partial x}(t, X_t^{t_0, x_0})\big)^2\Big)\ dt \tag{13.4}$$

be finite. Then

$$u(t_0, x_0) = E\Big(\varphi(X_T^{t_0, x_0}) - \int_{t_0}^{T} g(t, X_t^{t_0, x_0})\ dt\Big). \tag{13.5}$$

Of course, we can start our diffusion process at some other time, from some other point – and this will be shown in the superscript. In particular, we can start it from the point x at time t; only we'll no longer be allowed to use the letter t to denote the current time variable: we'll be speaking of the process $X_s^{t, x}$, $s \ge t$; and formula (13.5) becomes a formula for the solution at an arbitrary point (t, x):

$$u(t, x) = E\Big(\varphi(X_T^{t, x}) - \int_{t}^{T} g(s, X_s^{t, x})\ ds\Big). \tag{13.6}$$

Now let us look at the particular case of equations with the diffusion coefficient $a(t,x) \equiv 0$. Let us write the corresponding “degenerate parabolic” equation, taking, for simplicity's sake, $g(t,x) \equiv 0$:

$$\frac{\partial u}{\partial t}(t,x) + b(t,x) \cdot \frac{\partial u}{\partial t}(t,x) = 0. \tag{13.7}$$

This is just a linear first-order partial differential equation. It can be considered as a “degenerate parabolic” equation with the coefficient $\alpha(t,x) \geq 0$, for which an *initial* condition $u(t_0,x) = \varphi(x)$ should be imposed, to solve it for the values of $t \geq t_0$; or as one with $\alpha(t,x) \leq 0$: one to be solved for the values of the time t *down* from the time at which the initial (final) condition is prescribed. It turns out that the equation (13.7) can be solved in both directions: both up from t_0 and down.

The solution $X_s^{t,\,x}$ of the initial-value problem

$$\frac{dX_s}{ds} = b(s, X_s), \qquad X_t = x \tag{13.8}$$

is defined both for $s < t$ and for $s > t$, and it is not a random function (provided the solution is unique); the conditions of finiteness of expectations (13.4) are satisfied automatically, and formula (13.7) with t_0 instead of T becomes

$$u(t,x) = \varphi(X_{t_0}^{t,\,x}). \tag{13.9}$$

All this was under the assumption that a smooth solution $u(t,x)$ of the equation (13.7) with the initial condition $u(t_0,x) = \varphi(x)$ exists; but does it? and for what class of initial conditions $\varphi(x)$? For a given function $\varphi(x)$, let us *define* the function of two variables $u(t,x)$ by formula (13.9), and let us look whether it really solves our initial-value problem.

Of course, the initial condition $u(t_0,x) = \varphi(x)$ is satisfied because, by definition, $X_{t_0}^{t_0,\,x} = x$. Now let us consider the equation itself.

If the function $\varphi(x)$ is differentiable, we can write:

$$\frac{\partial u}{\partial t}(t,x) = \varphi'(X_{t_0}^{t,x}) \cdot \frac{\partial X_{t_0}^{t,x}}{\partial t}, \qquad \frac{\partial u}{\partial x}(t,x) = \varphi'(X_{t_0}^{t,x}) \cdot \frac{\partial X_{t_0}^{t,x}}{\partial x}. \tag{13.10}$$

Differentiability of the solution $X_s^{t,\,x}$ with respect to the initial point (t,x) is proved in the theory of ordinary differential equations under the condition that the (continuous) coefficient $b(t,x)$ has a continuous partial derivative $\dfrac{\partial b}{\partial x}$. And then the equation (13.7) is satisfied; this can be proved, e. g., using formula (13.6) with t_0 instead of T and the unnecessary expectation sign dropped:

$$\varphi(X_{t_0}^{t,\,x}) = u(t,x) = \varphi(X_{t_0}^{t,\,x}) - \int_t^{t_0} \big[\frac{\partial u}{\partial t}(s, X_s^{t,\,x}) + b(s, X_s^{t,\,x}) \cdot \frac{\partial u}{\partial t}(s, X_s^{t,\,x})\big]\, ds, \tag{13.11}$$

the integral is equal to 0, so is the integrand.

It is pretty clear that if the initial condition $\varphi(x)$ is not differentiable, the equation (13.7) is not satisfied.

This method of solving first-order partial differential equations is called *the method of characteristics*: solutions of the equation (13.8) are called *characteristics* for the equation (13.7).

So in fact, our method for solving the problem (13.1) using solutions of the stochastic equation (13.3) is a generalization of the method of characteristics, with *random* characteristics.

Now we go back to non-degenerate parabolic equations. Our link between partial differential equations and stochastic equations works under the assumption of a smooth solution $u(t,x)$ existing. But *does it* exist? Under what conditions? And is the solution unique?

Can we use formula (13.6) to prove the existence, as we did in the case of the degenerate (first-order) equation (13.7)? To use this method, we should establish differentiability of the solution $X_s^{t,\,x}$ of our stochastic equation with respect to the initial point x. We cannot see now how this can be done, and anyway, this was not in the original papers by Itô and Gikhman. It turns out that it can be done, under the assumption that continuous derivatives $\dfrac{\partial b}{\partial x}$, $\dfrac{\partial \sigma}{\partial x}$ exist; but we don't need it: we can combine the methods from the theory of partial differential equations and those of the theory of stochastic processes. In the theory of partial differential equations there are many good existence theorems for parabolic equations. It turns out that, in contrast with the first-order (degenerate parabolic) equation (13.7) having no solution for a non-differentiable initial condition $\varphi(x)$, a non-degenerate parabolic equation has a solution for a much wider class of initial (final) conditions. So here the PDE-theoretic methods work better than the probabilistic ones.

What the methods of stochastic processes are good for is *uniqueness*. Formula (13.6) provides an "explicit" representation for every solution of the problem (13.1), so there cannot be two different solutions: both of them would be equal to the same expectation.

However, almost the first thing that we learn about the Cauchy problem for, say, the heat equation is that its solution *is not unique*: an example of a non-identically-zero function satisfying this equation and having zero initial condition is provided.

So how is it: on one hand, the solution is not unique; but formula (13.6) tells us that it is?

But now we remember that formula (13.6) was true only under the condition (13.4) (with (t_0, x_0) changed to (t, x)) of finiteness of some expectations. If the function $u(t,x)$ (or its partial derivative) grows too fast as $|x| \to \infty$, these expectations are infinite, and the uniqueness result does not hold.

Let us formulate, however, a uniqueness result – in the language of partial differential equations (but our proof will be based on the stochastic theory):

Theorem 13.1. *Let $u_1(t,x)$, $u_2(t,x)$ be two solutions of the final-value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x), & t_0 \le t \le T, \;\; x \in (-\infty, \infty), \\ u(T,x) = \varphi(x), & x \in (-\infty, \infty) \end{cases} \tag{13.12}$$

(remember that the time derivative at the ends of the time interval is understood as the one-sided derivative, and the partial derivatives are supposed to be continuous). *Let there exist constants K and C such that for all t and x*

$$\Big|\frac{\partial u_i}{\partial x}(t,x)\Big| \le K\, e^{Cx^2}. \tag{13.13}$$

Then $u_1(t,x) = u_2(t,x)$ for all $t \in [t_0, T]$, $x \in (-\infty, \infty)$.

Proof. The function $u(t,x)$ satisfies a condition similar to (13.13):

$$|u(t,x)| \le |u(t,0)| + \int_0^{|x|} e^{Cy^2}\, dy \le \max_{t_0 \le t \le T} |F(t,0)| + |x|\, e^{Cx^2} \le K_1 e^{C_1 x^2} \tag{13.14}$$

with some $K_1 \ge K$, $C_1 \ge C$.

The distribution of the random variable $Y = W_s^{t,\,x}$ is normal with parameters $(x,\, s-t)$, with density

$$p(y) = p(t,x,s,y) = \frac{1}{\sqrt{2\pi(s-t)}}\, e^{-(y-x)^2/2(s-t)}. \tag{13.15}$$

We are going to prove the uniqueness first in a smaller interval $[T-\Delta, T]$, where $\Delta = 1/10C_1$.

For a given $x \in \mathbb{R}$, we can write:

$$K_1 e^{C_1 y^2} = K_1 e^{C_1(x+(y-x))^2} \le K_1 e^{2C_1 x^2} \cdot e^{2C_1(y-x)^2}. \tag{13.16}$$

We have, for $t \in [T-\Delta, T]$:

$$\begin{aligned} E\big(|u_i(T, W_T^{t,x})|\big) &\le \int_{-\infty}^{\infty} K_1 e^{2C_1 x^2} \cdot e^{2C_1(y-x)^2} \cdot \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-(y-x)^2/2(T-t)}\, dy \\ &= K_1 e^{2C_1 x^2} \cdot \frac{1}{\sqrt{2\pi(T-t)}} \cdot \sqrt{2\pi/\big(\frac{1}{T-t} - 4C_1\big)} < \infty, \end{aligned} \tag{13.16}$$

$$\begin{aligned} E\Big(\big(\frac{\partial u_i}{\partial x}(s, W_s^{t,x})\big)^2\Big) &\le \int_{-\infty}^{\infty} K_1^2 e^{4C_1 x^2} \cdot e^{4C_1(y-x)^2} \cdot \frac{1}{\sqrt{2\pi(s-t)}}\, e^{-(y-x)^2/2(s-t)}\, dy \\ &= K_1^2 e^{4C_1 x^2} \cdot \frac{1}{\sqrt{2\pi(s-t)}} \cdot \sqrt{2\pi/\big(\frac{1}{s-t} - 8C_1\big)} < \frac{K_1^2 e^{4C_1 x^2}}{\sqrt{1-8\Delta}}, \end{aligned} \tag{13.17}$$

$$\int_t^T E\Big(\big(\frac{\partial u_i}{\partial x}(s, W_s^{t,x})\big)^2\Big)\, ds < \frac{K_1^2 e^{4C_1 x^2}}{\sqrt{1-8\Delta}} \cdot \Delta < \infty. \tag{13.18}$$

So we can apply formula (13.6):

$$u_1(t,x) = E\Big(\varphi(W_T^{t,\,x}) - \int_t^T g(s, W_s^{t,\,x})\, ds\Big) = u_2(t,x). \tag{13.19}$$

Now for $T-2\Delta \le t \le T-\Delta$ we can use the same with $T-\Delta$ instead of T:

$$u_1(t,x) = E\Big(u_i(T-\Delta, W_{T-\Delta}^{t,\,x}) - \int_t^{T-\Delta} g(s, W_s^{t,\,x})\,ds\Big) = u_2(t,x); \tag{13.20}$$

etc. by intervals $[T-k\Delta, T-(k-1)\Delta]$ of the same length until we reach the time t_0.

As a matter of fact, in the theory of partial differential equations they do not require that the solution be smooth up to, and including, the time at which the initial (the final) condition is prescribed; the standard formulation of the Cauchy problem is:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -L_t u(t,x) + g(t,x), & t < T,\ \ x \in (-\infty,\infty), \\ u(T,x) = \varphi(x), & x \in (-\infty,\infty), \end{cases} \tag{13.21}$$

where the function $u(t,x)$ is supposed to be continuous for $t \le T$. (If we do not require continuity including the time point t_0, we can attach to the final condition $u(T,x) = \varphi(x)$ a function $u(t,x)$, $t < T$, having nothing to do with this condition, e. g., $u(t,x) \equiv 0$. Another proper way of imposing a final condition is requiring that $\lim_{t\to T^-,\,y\to x} u(t,y) = \varphi(x)$.)

I don't want to bother with functions that may go to infinity as $|x| \to \infty$, but not too fast, so I'll give some of the subsequent results under conditions of boundedness.

Theorem 13.2. *The solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\dfrac{1}{2}\,\dfrac{\partial^2 u}{\partial x^2}(t,x) + g(t,x), & t < T,\ \ x \in (-\infty,\infty), \\ u(T,x) = \varphi(x), & x \in (-\infty,\infty), \end{cases} \tag{13.22}$$

is unique in the class of functions $u(t,x)$ that are bounded and such that for every positive δ the partial derivative $\dfrac{\partial u}{\partial x}(t,x)$ is bounder for $t \le T-\delta$ (for each $\delta > 0$, by its own constant).

Proof. We can apply the previous theorem for $t \le T-\delta$, with the final condition prescribed at $t = T-\delta$, where δ is an arbitrary positive number that is less than $T-t$ ((t,x) being the point at which we want to prove our uniqueness); then, for two solutions $u_1(t,x)$ and $u_2(t,x)$, we get:

$$u_1(t,x) - u_2(t,x) = E\big(u_1(T-\delta, W_{T-\delta}^{t,\,x}) - u_2(T-\delta, W_{T-\delta}^{t,\,x})\big). \tag{13.23}$$

The random variable under the sign of expectation is dominated by the constant $\sup|u_1(t,x)| + \sup|u_2(t,x)|$. As $\delta \to 0^+$, this difference goes to $u_1(T, W_T^{t,\,x}) - u_2(T, W_T^{t,\,x}) = 0$; so by the dominated-convergence theorem (Theorem 2.2) we have that the right-hand side in (13.23) converges to 0. The left-hand side does not depend on δ, so it is just equal to 0.

Of course, we have not used here the fact that our diffusion process is a Wiener one; a more general result takes place:

Theorem 13.2′. *Let the diffusion coefficient* $a(t,x)$ *in the operator* L_t *satisfy the inequality* $a(t,x) \le A(x)$. *Then the solution of the Cauchy problem* (13.21) *is unique in the class of functions* $u(t,x)$ *that are bounded and such that* $\left|\dfrac{\partial u}{\partial x}(t,x)\right| \le C(\delta)/\sqrt{A(x)}$ *for* $t \le T - \delta$ *for every* $\delta > 0$.

Now to the existence.

Theorem 13.3. *Let* $\varphi(x)$ *be a bounded function. then there exists a bounded function* $u(t,x)$, $t < T$, *satisfying the equation* $\dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = 0$, *with the partial derivative* $\dfrac{\partial u}{\partial x}$ *that is bounded, for each positive* δ, *for* $t \le T - \delta$ (for each $\delta > 0$ by its own constant), *such that*

$$\lim_{t \to T^-,\, y \to x} u(t,y) = \varphi(x) \tag{13.24}$$

for every x *that is a continuity point of the final condition* φ.

Proof. Let us try to construct the solution using formula (13.6). Using the density (13.15), we can write:

$$u(t,x) = \int_{-\infty}^{\infty} \varphi(y) \cdot p(t,x,T,y)\, dy = \int_{-\infty}^{\infty} \varphi(y) \cdot \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-(y-x)^2/2(T-t)}\, dy. \tag{13.25}$$

The derivatives of this function are:

$$\frac{\partial u}{\partial t}(t,x) = \int_{-\infty}^{\infty} \varphi(y) \cdot \frac{\partial}{\partial t} p(t,x,T,y)\, dy, \tag{13.26}$$

$$\begin{aligned} \frac{\partial u}{\partial x}(t,x) &= \int_{-\infty}^{\infty} \varphi(y) \cdot \frac{\partial}{\partial x} p(t,x,T,y)\, dy \\ &= \int_{-\infty}^{\infty} \varphi(y) \cdot \frac{y-x}{\sqrt{2\pi(T-t)^3}}\, e^{-(y-x)^2/2(T-t)}\, dy, \end{aligned} \tag{13.27}$$

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \int_{-\infty}^{\infty} \varphi(y) \cdot \frac{\partial^2}{\partial x^2} p(t,x,T,y)\, dy. \tag{13.28}$$

Honest differentiation and a little algebra convince us that the equation $\dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = 0$ is satisfied; the x-derivative (13.27) is bounded in absolute value, for $t \le T - \delta$, by

$$\sup|\varphi(x)| \cdot \int_{-\infty}^{\infty} \frac{|x-y|}{\sqrt{2\pi(T-t)^3}}\, e^{-(y-x)^2/2(T-t)}\, dy = \sup|\varphi(x)| \cdot \frac{\sqrt{2/\pi}}{\sqrt{T-t}} \le \sup|\varphi(x)| \cdot \frac{\sqrt{2/\pi}}{\sqrt{\delta}}. \tag{13.29}$$

Now to the initial condition. If x is a continuity point of the function φ, for every $\varepsilon > 0$ there exists a positive δ such that

$$|z - x| < \delta \;\Rightarrow\; |\varphi(z) - \varphi(x)| < \frac{\varepsilon}{2}. \tag{13.30}$$

We have:

$$
\begin{aligned}
|u(t,y)-\varphi(x)| &= \Big|\int_{-\infty}^{\infty}\big(\varphi(z)-\varphi(x)\big)\cdot p(t,y,T,z)\;dz\Big| \\
&\le \int\limits_{(x-\delta,\,x+\delta)} |\varphi(z)-\varphi(x)|\cdot p(t,y,T,z)\;dz \\
&\qquad + \int\limits_{\mathbb{R}\setminus(x-\delta,\,x+\delta)} |\varphi(z)-\varphi(x)|\cdot p(t,y,T,z)\;dz.
\end{aligned} \tag{13.31}
$$

The first integral is less than $\varepsilon/2$; the second one does not exceed

$$
2\sup|\varphi(x)|\cdot\Big[1-\int_{x-\delta}^{x+\delta} p(t,y,T,z)\;dz\Big] = 2\sup|\varphi(x)|\cdot\big[1-\Phi\big(\frac{x+\delta-y}{\sqrt{T-t}}\big)+\Phi\big(\frac{x-\delta-y}{\sqrt{T-t}}\big)\big]. \tag{13.32}
$$

If $|y-x|<\delta/2$, the right-hand side is not greater than

$$
2\sup|\varphi(x)|\cdot\big[1-\Phi\big(\frac{\delta/2}{\sqrt{T-t}}\big)+\Phi\big(\frac{-\delta/2}{\sqrt{T-t}}\big)\big], \tag{13.33}
$$

which goes to 0 as $t\to T^-$. So we can choose a $\delta_1>0$ so that the expression (13.33) is less than $\varepsilon/2$ for $t>T-\delta_1$. Taking this together with $\varepsilon/2$ estimating the first integral in (13.31), we obtain $|u(t,y)-\varphi(x)|<\varepsilon$ for $|y-x|<\delta/2$, $t>T-\delta_1$; that is, (13.24).

Formula (13.25) is well known in the theory of partial differential equations. For parabolic equations other than the heat equation, representations of the form (13.25) are also proved, but without $p(t,x,s,y)$ being the normal density. The function $p(t,x,s,y)$ of four arguments is called *the fundamental solution.* So, from the point of view of stochastic processes, the fundamental solution is, as a function of its last argument y, the probability density of the value $X_s^{t,x}$ at the time point s of the diffusion process starting from the point x at time t $(<s)$.

If you solve Problem **13**, and write the corresponding probability density – you have found the fundamental solution for the parabolic equation $\dfrac{\partial u}{\partial t}+\dfrac{a}{2}\dfrac{\partial^2 u}{\partial x^2}-\mu x\,\dfrac{\partial u}{\partial x}=0$.

One more uniqueness theorem:

Theorem 13.4. *Let x_0 be some point in the real line. The solution of the problem*

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x) = -\dfrac{1}{2}\,\dfrac{\partial^2 u}{\partial x^2}(t,x)+g(t,x), & t<T,\quad x\in(-\infty,\infty),\\
u(T,x)=\varphi(x), & x\ne x_0,
\end{cases} \tag{13.34}
$$

(the Cauchy problem with the final condition satisfied except at one point) *is unique in the class of functions $u(t,x)$ that are bounded and such that for every positive δ the partial derivative $\dfrac{\partial u}{\partial x}(t,x)$ is bounder for $t\le T-\delta$.*

Without the requirement of boundedness, the statement is not true, as the following example shows:

$$u(t,x) = \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-(x-x_0)^2/2(T-t)}. \tag{13.35}$$

The **proof** is, practically, the same as that of Theorem 13.2, with applying the same dominated-convergence theorem to the almost-sure limit

$$\lim_{\delta\to 0+} u(T-\delta, X_{T-\delta}^{t,\,x}) = \varphi(X_T^{t,\,x}) \tag{13.36}$$

(this limit holds if $X_T^{t,\,x} \neq x_0$, and the probability $P\{X_T^{t,\,x} = x_0\} = 0$ because the normal distribution is a continuous one).

The statement of the theorem holds also if the final condition is satisfied except at finitely many points, or even except a countable set of points. It holds also for diffusion processes corresponding to parabolic, or degenerate parabolic equations, if their distributions have a density (or, in the language of differential equations: if there is a fundamental solution).