MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Lecture note 16. Conditional expectations and conditional probabilities with respect to several random variables.

The next thing: we go to the conditional expectations

$$E\{X|Y_1=y_1,...,Y_n=y_n\}, \qquad E(X\|Y_1,...,Y_n) \tag{16.1}$$

of a random variable X with respect to n random variables Y_1, ..., Y_n. By definition,

$$E\{X|Y_1=y_1,...,Y_n=y_n\}=\varphi(y_1,...,y_n), \qquad E(X\|Y_1,...,Y_n)=\varphi(Y_1,...,Y_n), \tag{16.2}$$

where the function φ is such that for every set $C\subseteq\mathbb{R}^n$

$$E\big(I_C(Y_1,...,Y_n)\cdot X\big)=E\big(I_C(Y_1,...,Y_n)\cdot\varphi(Y_1,...,Y_n)\big). \tag{16.3}$$

There is practically no difference with the case of one conditioning random variable Y: we can consider a *generalized* random variable (random vector) $\boldsymbol{Y}=(Y_1,...,Y_n)$, and use the same formulas, only with the boldface letter $\boldsymbol{Y}$. In particular, if the joint distribution of the random variables Y_1, ..., Y_n, X is a continuous one, with joint density $p_{Y_1,\,...,\,Y_n,\,X}(y_1,...,y_n,x)$, the conditional expectations are given by

$$E\{X|Y_1=y_1,...,Y_n=y_n\}=\int_{-\infty}^{\infty}x\cdot p_{X|Y_1=y_1,\,...,\,Y_n=y_n}(x)\,dx, \tag{16.4}$$

$$E\{f(X)|Y_1=y_1,...,Y_n=y_n\}=\int_{-\infty}^{\infty}f(x)\cdot p_{X|Y_1=y_1,\,...,\,Y_n=y_n}(x)\,dx, \tag{16.5}$$

$$E\{f(X,Y_1,...,Y_n)|Y_1=y_1,...,Y_n=y_n\}=\int_{-\infty}^{\infty}f(x,y_1,...,y_n)\cdot p_{X|Y_1=y_1,\,...,\,Y_n=y_n}(x)\,dx, \tag{16.6}$$

where the conditional density

$$p_{X|Y_1=y_1,\,...,\,Y_n=y_n}(x)=\frac{p_{Y_1,\,...,\,Y_n,\,X}(y_1,...,y_n,x)}{p_{Y_1,\,...,\,Y_n}(y_1,...,y_n)} \tag{16.7}$$

(for y_1, ..., y_n with $p_{Y_1,...,Y_n}(y_1,...,y_n)=0$, the conditional density and the conditional expectations (16.4) – (16.6) are defined in an arbitrary way).

Example 16.1. Let W_t, $t\ge t_0$, be a Wiener process starting, at time t_0, from the point x_0. For $t_0<t_1<...<t_n<s$, find the conditional expectation

$$E\big(f(W_s)\|W_{t_0},W_{t_1},...,W_{t_n}\big), \tag{16.8}$$

where $f(x)$ is a bounded function.

The random variable W_{t_0} is identically equal to x_0, so we need only find

$$E\{f(W_s)|W_{t_0}=x_0, W_{t_1}=x_1, ..., W_{t_n}=x_n\} \tag{16.9}$$

for this fixed value x_0 and for arbitrary values of x_1, ..., $x_n \in \mathbb{R}$.

The event $\{W_{t_0}=x_0\}$ is sure to occur (is $=\Omega$), so we can delete the mention of this from the condition (this part of the condition occurs for all sample points ω). We find the conditional density using formula (16.7) (denoting the argument of the density with the letter z to avoid confusion with the variables x_1, ..., x_n):

$$\begin{aligned} &p_{W_s|W_{t_1}=x_1,\dots W_{t_n}=x_n}(z) \\ &\quad = \frac{p_{t_1-t_0}(x_1-x_0)\cdot p_{t_2-t_1}(x_2-x_1)\cdot\ldots\cdot p_{t_n-t_{n-1}}(x_n-x_{n-1})\cdot p_{s-t_n}(z-x_n)}{p_{t_1-t_0}(x_1-x_0)\cdot p_{t_2-t_1}(x_2-x_1)\cdot\ldots\cdot p_{t_n-t_{n-1}}(x_n-x_{n-1})}, \end{aligned} \tag{16.10}$$

where for $t>0$, $u\in(-\infty,\infty)$

$$p_t(u)=\frac{1}{\sqrt{2\pi t}}\,e^{-u^2/2t}. \tag{16.11}$$

Almost all factors in (16.10) cancel, and we obtain:

$$p_{W_s|W_{t_1}=x_1,\dots W_{t_n}=x_n}(z)=p_{s-t_n}(z-x_n)=\frac{1}{\sqrt{2\pi(s-t_n)}}\,e^{-(z-x_n)^2/2(s-t_n)}. \tag{16.12}$$

So we get:

$$E\{f(W_s)|W_{t_0}=x_0, W_{t_1}=x_1, ..., W_{t_n}=x_n\}=\int_{-\infty}^{\infty} f(z)\cdot p_{s-t_n}(z-x_n)\,dz. \tag{16.13}$$

The conditional expectation (16.8) is obtained by replacing x_0, x_1, ..., x_{n-1}, x_n with W_{t_0}, W_{t_1}, ..., $W_{t_{n-1}}$, W_{t_n} (but the function (16.13) does not depend from the variables x_0, x_1, ..., x_{n-1}):

$$E\big(f(W_s)\|W_{t_0}, W_{t_1}, ..., W_{t_n}\big)=\int_{-\infty}^{\infty} f(z)\cdot\frac{1}{\sqrt{2\pi(s-t_n)}}\,e^{-(z-W_{t_n})^2/2(s-t_n)}\,dz. \tag{16.14}$$

In particular, for $C\subseteq\mathbb{R}$ the conditional probability

$$\begin{aligned} P\{W_s\in C\|W_{t_0}, W_{t_1}, ..., W_{t_n}\} &= E\big(I_C(W_s)\|W_{t_0}, W_{t_1}, ..., W_{t_n}\big) \\ &= \int_C \frac{1}{\sqrt{2\pi(s-t_n)}}\,e^{-(z-W_{t_n})^2/2(s-t_n)}\,dz. \end{aligned} \tag{16.15}$$