MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Lecture note 19. Conditional expectations and conditional probabilities with respect to random variables. Properties, continued.

Continuation of what was started in the previous lecture note.

What exactly are we to prove? We claim that the right-hand side of (18.28) or (18.29) is the conditional expectation of the product $X \cdot Z$ with respect to Y_t, $t \in T_0$ (formula (18.30) is just the standard reformulation of this in another language). This means, as we know, two things: the first one, that this right-hand side is a function(al) of the random variables Y_t, $t \in T_0$; this is true, because the random variable Z is a function(al) of Y_t, $t \in T_0$, and the conditional expectation $E(X \| Y_t,\, t \in T_0)$ also is a function(al) (another one) of the same random variables, so their product also is one.

The second thing is that for an arbitrary event B associated with the random variables Y_t, $t \in T_0$, – that is, for an arbitrary set C in the appropriate space and the event $B = \{(Y_t,\, t \in T_0) \in C\}$ – we must have:

$$E(I_B \cdot XZ) = E\big(I_B \cdot Z E(X \| Y_t,\, t \in T_0)\big); \tag{19.1}$$

or, expanding the shortened, one-letter notations:

$$E\big(I_C(Y_t,\, t \in T_0) \cdot f(Y_t,\, t \in T_0) \cdot X\big) = E\big(I_C(Y_t,\, t \in T_0) \cdot f(Y_t,\, t \in T_0) \cdot E(X \| Y_t,\, t \in T_0)\big). \tag{19.2}$$

Formula (19.2) is a particular case of the following (shorter, but more general) formula:

$$E\big(g(Y_t,\, t \in T_0) \cdot X\big) = E\big(g(Y_t,\, t \in T_0) \cdot E(X \| Y_t,\, t \in T_0)\big), \tag{19.3}$$

where g is an arbitrary function(al). The particular case of (19.2) is obtained by taking $g = I_C \cdot f$.

The proof of formula (19.3) (don't be afraid, I won't give the complete proof here, I only wanted you to understand *what* was to be proved) is carried out by approximating general random variables by discrete ones taking finitely many values: the standard trick in the theory of measure and integration.

Let us only prove (19.3) in the case that the function(al) $g(y_t,\, t \in T_0)$ takes finitely many values g^1, g^2, ..., g^k. Let $C_i = \{y_\bullet \colon\ g(y_\bullet) = g^i\}$; then

$$\begin{aligned} E\big(g(Y_t,\, t \in T_0) \cdot X\big) &= \sum_{i=1}^{k} g^i \cdot E\big(I_{C_i}(Y_t,\, t \in T_0) \cdot X\big) \\ &= \sum_{i=1}^{k} g^i \cdot E\big(I_{C_i}(Y_t,\, t \in T_0) \cdot E(X \| Y_t,\, t \in T_0)\big) \\ &= E\big(g(Y_t,\, t \in T_0) \cdot E(X \| Y_t,\, t \in T_0)\big). \end{aligned} \tag{19.4}$$

As an application, let us prove

Microtheorem 19.1. *Let* $E(X^2) < \infty$. *Then also*

$$E\big(\big(E(X\|Y_t,\, t\in T_0)\big)^2\big) < \infty, \tag{19.5}$$

and

$$\|E(X\|Y_t,\, t\in T_0)\|_2 \le \|X\|_2. \tag{19.6}$$

Proof. Suppose at first that the expectation (19.5) is finite, and let us prove (19.6). We have:

$$\begin{aligned} E(X^2) &= E\big(\big(E(X\|Y_t,\, t\in T_0) + \big(X - E(X\|Y_t,\, t\in T_0)\big)\big)^2\big) \\ &= E\big(\big(E(X\|Y_t,\, t\in T_0)\big)^2\big) + 2E\big(E(X\|Y_t,\, t\in T_0)\cdot\big(X - E(X\|Y_t,\, t\in T_0)\big)\big) \\ &\qquad\qquad + E\big(\big(X - E(X\|Y_t,\, t\in T_0)\big)^2\big). \end{aligned} \tag{19.7}$$

The first term in the right-hand side is $\|E(X\|Y_t,\, t\in T_0)\|_2^2$; the third term is clearly nonnegative. Apply to the expectation of the product the formula (18.3) (the generalized Total Expectation Formula):

$$\begin{aligned} &E\big(E(X\|Y_t,\, t\in T_0)\cdot\big(X - E(X\|Y_t,\, t\in T_0)\big)\big) \\ &\qquad = E\big(E\big(E(X\|Y_t,\, t\in T_0)\cdot\big(X - E(X\|Y_t,\, t\in T_0)\big)\big\|Y_t,\, t\in T_0\big)\big). \end{aligned} \tag{19.8}$$

In the product under the sign of the conditional expectation, the first factor is a functional of Y_t, $t\in T_0$; so it can be taken out:

$$\begin{aligned} &E\big(E(X\|Y_t,\, t\in T_0)\cdot\big(X - E(X\|Y_t,\, t\in T_0)\big)\big\|Y_t,\, t\in T_0\big) \\ &\qquad = E(X\|Y_t,\, t\in T_0)\cdot E\big(X - E(X\|Y_t,\, t\in T_0)\big\|Y_t,\, t\in T_0\big). \end{aligned} \tag{19.9}$$

The remaining conditional expectation

$$\begin{aligned} &E\big(X - E(X\|Y_t,\, t\in T_0)\big)\big\|Y_t,\, t\in T_0\big) \\ &\qquad = E(X\|Y_t,\, t\in T_0) - E\big(E(X\|Y_t,\, t\in T_0)\big\|Y_t,\, t\in T_0\big) \\ &\qquad = E(X\|Y_t,\, t\in T_0) - E(X\|Y_t,\, t\in T_0) = 0, \end{aligned} \tag{19.10}$$

because taking twice the conditional expectation with respect to the same random variables is the same as taking it once (see Property 1)).

So the second term in the right-hand side of (19.7) is equal to 0, and we have $\|X\|_2^2 \ge \|E(X\|Y_t,\, t\in T_0)\|_2^2$.

Now, I have already skipped some part of proof having to do with approximating with discrete random variables from the proof of Property 4); so I don't want to skip the same kind of reasoning here.

First of all, for X being a discrete random variable taking finitely many values x^1, x^2, ..., x^k (19.5) is true, because

$$\big(E(X\|Y_t,\, t\in T_0)\big)^2 \le \big(\max(|x^1|, ..., |x^k|)\big)^2. \tag{19.11}$$

The next step: it is enough to prove it only for nonnegative random variables. Indeed, clearly

$$X = X_+ - X_-, \qquad \text{where } X_+ = \begin{cases} X, & X \geq 0, \\ 0, & X < 0, \end{cases} \quad X_+ = \begin{cases} 0, & X \geq 0, \\ -X, & X < 0; \end{cases} \tag{19.12}$$

$E(X_+^2)$, $E(X_-^2) \leq E(X^2) < \infty$.

Now, every nonnegative random variable X can be approximated with nonnegative discrete random variables X_n taking each finitely many values, and with $0 \leq X_n \leq X$; e. g., we can take

$$X_n(\omega) = \sum_{i=0}^{n \cdot 2^n - 1} \frac{1}{2^n} \cdot I_{\{i/2^n \leq X < (i+1)/2^n\}}(\omega) \tag{19.13}$$

(draw a picture, in which the sample space will be on the horizontal axis, and the nonnegative random variable $X(\omega)$ (unbounded!) on the vertical; show in your picture the discrete random variables $X_1(\omega)$, $X_2(\omega)$). It is clear that the random variables X_n form a non-decreasing sequence: $0 \leq X_1 \leq X_2 \leq X_n \leq \ldots$, $\lim_{n \to \infty} X_n(\omega) = X(\omega)$ for every $\omega \in \Omega$.

Of course, I forgot to mention some *very* simple and natural properties of the conditional expectation: that it is linear:

$$E(c_1 \cdot X_1 + c_2 \cdot X_2 \| Y_1,\, t \in T_0) = c_1 \cdot E(X_1 \| Y_t,\, t \in T_0) + c_2 \cdot E(X_2 \| Y_t,\, t \in T_0); \tag{19.14}$$

or that if $X_1 \leq X_2$, then (almost surely, of course)

$$E(X_1 \| Y_t,\, t \in T_0) \leq E(X_2 \| Y_t,\, t \in T_0). \tag{19.15}$$

So for our nondecreasing sequence almost surely

$$E(X_1 \| Y_t,\, t \in T_0) \leq E(X_2 \| Y_t,\, t \in T_0) \leq \ldots \leq E(X_n \| Y_t,\, t \in T_0) \leq \ldots. \tag{19.16}$$

So almost surely exists $\lim_{n \to \infty} E(X_n \| Y_t,\, t \in T_0)$. It seems clear that this limit *is* the conditional expectation $E(X \| Y_t,\, t \in T_0)$; but we are in a mood for proving "obvious" things: so, what are we to prove here?

Obviously, the limit of function(al)s of Y_t, $t \in T_0$, is also a function(al) of the same random variables. So we have to check that for an event B of the form $B = \{Y_\bullet \in C\}$ we have:

$$E(I_B \cdot X) = E\big(I_B \cdot \lim_{n \to \infty} E(X_n \| Y_t,\, t \in T_0)\big). \tag{19.17}$$

This follows from the monotone-convergence theorem (Theorem 2.1).

Now to the expectations of the squares. Clearly,

$$E\big((E(X_n \| Y_t,\, t \in T_0))^2\big) \leq E(X_n^2) \leq E(X^2) \tag{19.18}$$

(the question of finiteness of this expectation does not arise, because the random variable in it is bounded by some constant – the constant n^2 if we define the approximating discrete random variables by (19.13)).

Now, the squares of the conditional expectations form also a non-decreasing sequence:

$$\big(E(X_1\|Y_t,\, t \in T_0)\big)^2 \le \big(E(X_2\|Y_t,\, t \in T_0)\big)^2 \le ... \le \big(E(X_n\|Y_t,\, t \in T_0)\big)^2 \le ..., \tag{19.19}$$

and by the same theorem we have:

$$\begin{aligned} E\big(\big(E(X\|Y_t,\, t \in t_0)\big)^2\big) &= E\big(\lim_{n\to\infty}\big(E(X_n\|Y_t,\, t \in T_0)\big)^2\big) \\ &= \lim_{n\to\infty} E\big(\big(E(X_n\|Y_t,\, t \in T_0)\big)^2\big) \le \lim_{n\to\infty} E(X_n^2) \le E(X^2). \end{aligned} \tag{19.20}$$

Microtheorem 19.2. *If* $\text{l.i.m.}_{n\to\infty} X_n = Z$, *then* $\text{l.i.m.}_{n\to\infty} E(X_n\|Y_t,\, t \in T_0) = E(Z\|Y_t,\, t \in T_0)$ (provided the conditional expectations exist).

Proof. By the previous microtheorem, we have:

$$\|E(X_n\|Y_t,\, t \in T_0) - E(Z\|Y_t,\, t \in T_0)\|_2 = \|E(X_n - Z\|Y_t,\, t \in T_0)\|_2 \le \|X_n - Z\|_2 \to 0 \tag{19.21}$$

as $n \to \infty$.