Lecture note 21. Diffusion processes as Markov processes.

Another class of examples of Markov processes is diffusion processes, in the case that the stochastic equation defining them

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \qquad X_{t_0} = x_0 \tag{21.1}$$

has a unique solution.

How is this proved (for definiteness, let it be a one-dimensional diffusion process)?

We have to prove that for every t_0, x_0, for every $t \geq t_0$ and $s > t$, and for every $C \subseteq \mathbb{R}$

$$P\{X_s^{t_0,\,x_0} \in C \| X_u^{t_0,\,x_0},\, t_0 \leq u \leq t\} = \mu_{t,\,X_t^{t_0,\,x_0},\,s}(C), \tag{21.2}$$

where $\mu_{t,\,x,\,s}(C)$ is some function of $t < s$, $x \in \mathbb{R}$, and $C \subseteq \mathbb{R}$: the transition function.

If we replace in (21.2) t_0 with t, and x_0 with x, the conditioning random variables reduce to one, X_t, and that one identically equal to the prescribed initial point x; so we have:

$$\mu_{t,\,x,\,s}(C) = P\{X_s^{t,\,x} \in C\}. \tag{21.3}$$

So what we have to prove is that

$$P\{X_s^{t_0,\,x_0} \in C \| X_u^{t_0,\,x_0},\, t_0 \leq u \leq t\} = P\{X_s^{t,\,x} \in C\}\Big|_{x = X_t^{t_0,\,x_0}}. \tag{21.4}$$

We obtain $X_s^{t_0,\,x_0}$ by solving our stochastic equation for the times $\geq t_0$ with the initial condition $X_{t_0}^{t_0,\,x_0} = x_0$; but we can also obtain it solving the same stochastic equation for times $\geq t$, with the initial condition $X_t^{t_0,\,x_0}$ at time t. This can be written as

$$X_s^{t_0,\,x_0} = X_s^{t,\,x}\Big|_{x = X_t^{t_0,\,x_0}}. \tag{21.5}$$

For $x \in \mathbb{R}$, the random variable $X_s^{t,\,x}$ is a functional of the Wiener process W_u on the time interval from t to s; what is more, it depends only on the differences $W_u - W_t$, $t \leq u \leq s$. This is because in all stochastic integrals we handle only differences $W_{t_i} - W_{t_{i-1}}$. So we have:

$$X_s^{t,\,x} = f(t,\,x,\,s;\,W_u - W_t,\,t \leq u \leq s), \tag{21.6}$$

f being some functional.

The random variable $X_t^{t_0,\,x_0}$ is also represented as some functional of the Wiener process, but on the time interval from t_0 to t:

$$X_t^{t_0,\,x_0} = g(t_0,\,x_0,\,t;\,W_u,\,t_0 \leq u \leq t) \tag{21.7}$$

(we could also have written it as a functional of the differences $W_u - W_{t_0}$). So we can write:

$$X_s^{t_0,\,x_0} = f\big(t,\,g(t_0,\,x_0,\,t; W_u,\,t_0 \leq u \leq t),\,s;\,W_u - W_t,\,t \leq u \leq s\big). \tag{21.8}$$

The conditional probability (21.4) is written as

$$P\{f\bigl(t,\, g(t_0,\, x_0,\, t;\, W_u,\, t_0 \le u \le t),\, s;\, W_u - W_t,\, t \le u \le s\bigr) \in C \| X_u,\, t_0 \le u \le t\}$$
$$= E\bigl(h\bigl(g(t_0,\, x_0,\, t;\, W_u,\, t_0 \le u \le t),\, W_u - W_t,\, t \le u \le s\bigr) \| X_u,\, t_0 \le u \le t\bigr), \tag{21.9}$$

where

$$h(x,\, W_u - W_t,\, t \le u \le s\bigr) = I_C\bigl(f(t,\, x,\, s; W_u - W_t,\, t \le u \le s)\bigr). \tag{21.10}$$

Now, the two random objects (two collections of random variables) W_u, $t_0 \le u \le t$, and $W_u - W_t$, $t \le u \le t$, are independent; and we can use the following

Lemma. *If the random objects* Y, Z *are independent*, g *an arbitrary function*, h *a bounded one, then*

$$E\bigl(h\bigl(g(Y), Z\bigr) \| Y\bigr) = E\bigl(h(x, Z)\bigr)\big|_{x=g(Y)}. \tag{21.11}$$

This proves (21.4).