

Lecture note 23. Martingales.

A second class of stochastic processes I would like to speak about is *martingales*. To be precise, I would rather speak of this class as that of *random functions*: the standard way of *martingales* being used is as follows: We are interested in a stochastic process X_t ; starting from it, we construct an auxiliary random function $Y_t(\omega)$; it turns out to be a martingale – and then we can deduce from this fact some consequences *about the process* X_t . So martingales are of interest usually not in their own right – not as mathematical models for real-world processes exhibiting stochastic behavior, but rather as a tool to learn something about other stochastic processes.

Let $X_t, t \in T \subseteq \mathbb{R}$, be a stochastic process (taking values in some space SP). Let $Y_t, t \in T$, be a random function (taking *number* values). We say that Y_t is a *martingale* (with respect to the stochastic process X_t) if the following two conditions are satisfied:

1) for every $t \in T$ the random variable $Y_t = Y_t(\omega)$ is a functional of the values of the reference process X up to time t :

$$Y_t = Y_t(X_u, u \leq t); \tag{23.1}$$

2) for every couple of moments $t < s$ in T

$$Y_t = E(Y_s \| X_u, u \leq t) \tag{23.2}$$

(of course, we really should add: “almost surely”).

From the conditions 1), 2), the first condition is very simple: it even has nothing to do with any probabilities; 2) is more complicated.

We say that a random function $Z_t, t \in T$, is a *submartingale* if condition 1) is satisfied, and

3) for every couple of moments $t < s$ in T

$$Z_t \leq E(Z_s \| X_u, u \leq t). \tag{23.3}$$

Condition (23.2) can be rewritten in the form

$$E(Y_s - Y_t \| X_u, u \leq t) = 0, \tag{23.4}$$

or, deciphering what the conditional expectation is: for every set C in the space of functions $x_u, u \leq t$,

$$E(I_C(X_u, u \leq t) \cdot (Y_s - Y_t)) = 0. \tag{23.5}$$

Condition (23.3) defining a submartingale can be rewritten as

$$E(Z_s - Z_t \| X_u, u \leq t) \geq 0, \tag{23.6}$$

$$E(I_C(X_u, u \leq t) \cdot (Z_s - Z_t)) \geq 0. \quad (23.7)$$

Before we consider examples, let me formulate a result that facilitates working with martingales in the discrete-time case.

Microtheorem 23.1. *Let X_t, Y_t be random functions defined for some (finite or infinite) interval of integers $t: t \in T = \{(\dots), m, m+1, m+2, \dots, n-1, n, (\dots)\}$. Then $Y_t, t \in T$, is a martingale (a submartingale) with respect to X_t if and only if for every $t \in T$ such that $t+1 \in T$*

$$Y_t = E(Y_{t+1} \| X_u, u \leq t) \quad (\text{or } Z_t \leq E(Z_{t+1} \| X_u, u \leq t)). \quad (23.8)$$

Proof. We have to prove that not only (23.8) is satisfied, but also

$$Y_t = E(Y_{t+2} \| X_u, u \leq t) \quad (\text{or } Z_t \leq E(Z_{t+2} \| X_u, u \leq t)), \quad (23.9)$$

$$Y_t = E(Y_{t+3} \| X_u, u \leq t) \quad (\text{or } Z_t \leq E(Z_{t+3} \| X_u, u \leq t)), \quad (23.10)$$

etc. By Property 2) of the conditional expectation, we have:

$$E(Y_{t+2} \| X_u, u \leq t) = E(E(Y_{t+2} \| X_u, u \leq t+1) \| X_u, u \leq t). \quad (23.11)$$

By (23.8), we have:

$$E(Y_{t+2} \| X_u, u \leq t+1) = Y_{t+1} \quad (\text{or } E(Z_{t+2} \| X_u, u \leq t+1) \geq Z_{t+1}), \quad (23.12)$$

so the expectation (23.11) is equal to (or \geq)

$$E(Y_{t+1} \| X_u, u \leq t) = Y_t \quad (\text{or } \geq) \quad (23.13)$$

by (23.8). To prove (23.10), we insert two conditional expectations: with respect to $X_u, u \leq t+2$, and with respect to $X_u, u \leq t+1$; etc.

Formulas (23.8) can be rewritten as

$$E(Y_{t+1} - Y_t \| X_u, u \leq t) = 0 \quad (E(Z_{t+1} - Z_t \| X_u, u \leq t) \geq 0), \quad (23.14)$$

$$E(I_C(X_u, u \leq t) \cdot (Y_{t+1} - Y_t)) = 0 \quad (E(I_C(X_u, u \leq t) \cdot (Z_{t+1} - Z_t)) \geq 0). \quad (23.15)$$

Example 23.1. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent identically distributed random variables. Is this sequence a martingale (with respect to the same sequence as the reference sequence)?

By Property 2) of the conditional expectation, we have:

$$E(X_{n+1} \| X_k, k \leq n) = E(X_{n+1}) = \text{const}, \quad (23.16)$$

which is *not* almost surely equal to the random variable X_n (unless all our random variables take with probability one and the same constant value). So the sequence of X_n is not a

martingale. It is not a submartingale either, because X_n may be greater or smaller than the expectation (which is also the conditional expectation) $E(X_{n+1})$.

Example 23.2 (the first example of a random function which *is* a martingale). Let $X_1, X_2, \dots, X_n, \dots$ be independent random variables taking values ± 1 with probabilities $1/2$. For every n , let $f_n(x_1, \dots, x_{n-1})$ be a function of the arguments x_i taking values ± 1 (for $n = 1$, f_0 is a function of zero arguments, which is taken to mean a constant). Let us take a constant y_0 and form the sums

$$\begin{aligned} Y_0 &= y_0, & Y_1 &= y_0 + f_1 \cdot X_1, & Y_2 &= y_0 + f_1 \cdot X_1 + f_2(X_1) \cdot X_2, \\ & & & & Y_3 &= y_0 + f_1 \cdot X_1 + f_2(X_1) \cdot X_2 + f_3(X_1, X_2) \cdot X_3, \quad \dots, \\ & & & & Y_n &= y_0 + \sum_{i=1}^n f_i(X_1, \dots, X_{i-1}) \cdot X_i, \quad \dots \end{aligned} \tag{23.17}$$

Let us check that the sequence $Y_0, Y_1, Y_2, \dots, Y_n, \dots$ is a martingale with respect to the reference sequence $X_1, X_2, \dots, X_n, \dots$ (we have no random variable X_0 , but this is OK; if we want to, we can introduce $X_0 \equiv 0$).

The sequence $\{Y_n\}$ can be interpreted as follows: We toss a coin time after time, and bet a quantity f_i on the outcome of the i -th tossing. The size of our bet depends, in general, on the outcomes of the previous tosses (we would be too lucky if we knew X_n before assigning the size f_n of our bet; but we take it to be impossible). The quantity y_0 represents our initial capital, and Y_i our capital after n bets.

We have: $Y_0 \equiv y_0$ is a constant: a function of an empty collection of random variables X_i ; and clearly every $Y_n, n \geq 1$, is a function of X_1, \dots, X_n . Now we have to check that for every $C \subseteq \mathbb{R}^n$

$$E(I_C(X_1, \dots, X_n) \cdot (Y_{n+1} - Y_n) \mid X_1, \dots, X_n) = 0. \tag{23.18}$$

The random variable under the conditional expectation sign is equal to $I_C(X_1, \dots, X_n) \cdot f_{n+1}(X_1, \dots, X_n) \cdot X_{n+1}$; by Properties 4) and 2) of the conditional expectations we have:

$$\begin{aligned} & E(I_C(X_1, \dots, X_n) \cdot (Y_{n+1} - Y_n) \mid X_1, \dots, X_n) \\ &= I_C(X_1, \dots, X_n) \cdot f_{n+1}(X_1, \dots, X_n) \cdot E(X_{n+1} \mid X_1, \dots, X_n) \\ &= I_C(X_1, \dots, X_n) \cdot f_{n+1}(X_1, \dots, X_n) \cdot E(X_{n+1}) = 0. \end{aligned} \tag{23.19}$$

Generalization: if $X_1, X_2, \dots, X_n, \dots$ are independent random variables with *arbitrary* distributions with $E(X_i) = 0$, and $f_n(x_1, \dots, x_{n-1})$ arbitrary functions such that $E(|f_n(X_1, \dots, X_{n-1})|) < \infty$, the sequence (23.17) is a martingale: our capitals after participating n times in a *fair game* ($E(X_i) = 0$).

If $E(X_i) \geq 0$ (the game is favorable to us) and $f_n(x_1, \dots, x_{n-1}) \geq 0$ (we don't bet against our luck), the sequence (23.17) is a *submartingale*.

When J.L.Doob, the American mathematician who wrote the book "Stochastic Processes", in which much attention was paid to martingales, visited Moscow, we asked him why this class of random functions is called *martingales*: what does this word mean? He

explained that a martingale was a device checking upward movement of a horse's head: a strap fastened to the bit. This seemed to have nothing to do with the mathematical concept: or did it contain some depths that we couldn't grasp? But in fact, there is another meaning of the word "martingale": "any of several systems of betting in which a player increases his stake usually by doubling each time he loses a bet". The above example shows that, with a fair game, no system of betting can increase (or decrease) your expected winnings.

To be continued.