Lecture note 24. Martingales, continued.

Example 24.1. Suppose Y_t, $t \in T$, is a martingale with respect to X_t; and suppose $E(Y_t^2) < \infty$. Then the random function $Z_t = Y_t^2$, $t \in R$, is *a submartingale.*

Requirement 1) is very simple: Y_t is a functional of the values of the stochastic process X_u up to time t: $Y_t = Y_t(X_u,\, u \le t)$; clearly $Z_t = \big(Y_t(X_u,\, u \le t)\big)^2$ is also a functional of X_u, $u \le t$.

Requirement 3):

$$\begin{aligned} E(Y_s^2 \| X_u,\, u \le t) &= E\big(\big(Y_t + (Y_s - Y_t)\big)^2 \big\| X_u,\, u \le t\big) \\ &= E\big(Y_t^2 + 2Y_t(Y_s - Y_t) + (Y_s - Y_t)^2 \big\| X_u,\, u \le t\big) \\ &= E(Y_t^2 \| X_u,\, u \le t) + 2E\big(Y_t(Y_s - Y_t) \big\| X_u,\, u \le t\big) \\ &\qquad + E\big((Y_s - Y_t)^2 \big\| X_u,\, u \le t\big). \end{aligned} \tag{24.1}$$

The first summand is equal to Y_t^2 by Property 1) of conditional expectations. In the second, you can take Y_t from under its sign:

$$E\big(Y_t(Y_s - Y_t) \big\| X_u,\, u \le t\big) = Y_t \cdot E(Y_s - Y_t \| X_u,\, u \le t) = 0. \tag{24.2}$$

The third is nonnegative; so

$$E(Y_s^2 \| X_u,\, u \le t) \ge Y_t^2, \tag{24.3}$$

which was to be proved.

Example 24.1′. It can be proved that if Y_t is a martingale, a function $f(y)$ is concave upwards, then $f(Y_t)$ is a submartingale (provided $E\big(|f(Y_t)|\big) < \infty$). The proof uses the fact that for every y_0 there exists a constant $m(y_0)$ such that $f(y) \ge f(y_0) + m(y_0)\cdot(y - y_0)$ (for $f(y) = y^2$ this is the inequality $y^2 \ge y_0^2 + 2y_0 \cdot (y - y_0)$; if $f(y)$ is concave and *differentiable* at $y = y_0$, this is the fact that an upward concave function lies above its tangent line).

Example 24.2. A stochastic integral $Y_t = \int_{t_0}^{t} g(s, \omega)\, dW_s$ of a random function $g(s, \omega)$ determined by the past W_u, $t_0 \le u \le s$, of the Wiener process is a martingale.

Let us check this first for step random functions $g(s, \omega)$:

$$g(s, \omega) = \sum_{i=1}^{n} Z_i(W_u,\, u \le t_{i-1}) \cdot I_{(t_{i-1},\, t_i]}(s). \tag{24.4}$$

We have to check that for $t_0 \le t < s$

$$E\Big(\int_{t_0}^{s} f(v, \omega)\, dW_v - \int_{t_0}^{t} f(v, \omega)\, dW_v \Big\| W_u,\, u \le t\Big) = 0. \tag{24.5}$$

Without restricting generality, we can assume that $t = t_m < t_{m+1} < ... < t_n = s$ (if t and s do not belong to the partition points t_i, we add them to the partition points). We have to prove that

$$\begin{aligned} E\Big(\sum_{i=m+1}^{n} Z_i \cdot (W_{t_i} - W_{t_{i-1}}) \Big\| W_u,\, t_0 \le u \le t\Big) \\ = \sum_{i=m+1}^{n} E\big(Z_i \cdot (W_{t_i} - W_{t_{i-1}}) \big\| W_u,\, t_0 \le u \le t\big) = 0. \end{aligned} \tag{24.6}$$

In the i-th summand, let us use Property 3) of conditional expectations:

$$\begin{aligned} &E\big(Z_i \cdot (W_{t_i} - W_{t_{i-1}}) \big\| W_u,\, t_0 \le u \le t\big) \\ &\quad = E\big(E\big(Z_i \cdot (W_{t_i} - W_{t_{i-1}}) \big\| W_u,\, t_0 \le u \le t_{i-1}\big) \big\| W_u,\, t_0 \le u \le t\big) \\ &\quad = E\big(Z_i \cdot E(W_{t_i} - W_{t_{i-1}} \| W_u,\, t_0 \le u \le t_{i-1}) \big\| W_u,\, t_0 \le u \le t\big). \end{aligned} \tag{24.7}$$

By Property 2) of conditional expectations, the inner conditional expectation is equal to the unconditional one, $E(W_{t_i} - W_{t_{i-1}}) = 0$. So (24.5) is proved.

Now, the stochastic integral of an arbitrary random function is defined as the mean-square limit of those of step functions:

$$\int_t^s f(v,\omega)\, dW_v = \operatorname*{l.i.m.}_{\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0} \int_t^s f_{\mathfrak{T}}(v,\omega)\, dW_v. \tag{24.8}$$

By Microtheorem 19.2, we have:

$$\begin{aligned} &E\Big(\int_t^s f(v,\omega)\, dW_v \Big\| W_u\, u \le t\Big) \\ &\quad = \operatorname*{l.i.m.}_{\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0} E\Big(\int_t^s f_{\mathfrak{T}}(v,\omega)\, dW_v \Big\| W_u\, u \le t\Big) = \operatorname*{l.i.m.}_{\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0} 0 = 0. \end{aligned} \tag{24.9}$$

More theory of martingales, later.