Lecture note 25. Stopping times.

Let X_t, $t \in T \subseteq \mathbb{R}$, be a stochastic process (it will be our process of reference). A random variable $\tau = \tau(\omega)$ taking values in $T \cup \{\infty\}$ (that is, $\tau \in T$ or is possibly infinite) is called *a stopping time* if for every $t \in T$ the event $\{\tau \leq t\}$ is determined by the values of our stochastic process X_u for $u \leq t$: there exists a subset C in the space of functions x_u, $u \leq t$, such that

$$\{\omega\colon\, \tau(\omega) \leq t\} = \big\{\omega\colon\, \big(X_u(\omega),\, u \leq t\big) \in C\big\}. \tag{25.1}$$

That is, for every t we should be able, observing the reference process X_u up to this time, to determine whether the moment τ has already come ($\tau \leq t$) or not.

The name "stopping time" is used because of the use of this concept in the theory of martingales: we'll see that if we *stop* a martingale at a stopping random time τ, it still remains a martingale. There is another name for the same class of mathematical objects: *Markov times*: from their use in the theory of Markov processes.

The concept of stopping time has nothing to do with any probabilities or expectations; so it does not belong to the theory of stochastic processes properly speaking but rather to the set-theoretic introduction to it.

Example 25.1. Every constant $t_* \in T$, and also $\tau \equiv \infty$ is a Markov time.
Indeed,

$$\{\omega\colon\, t_* \leq t\} = \begin{cases} \Omega & \text{if } t_* \leq t, \\ \emptyset & \text{if } t_* > t. \end{cases} \tag{25.2}$$

Each of these events is represented as $\{\omega :\, (X_u,\, u \leq t) \in C\}$ with C being some subset in the space of functions x_u, $u \leq t$: the event Ω, with C being the whole space of these functions, with the impossible event $\emptyset$, with $C = \emptyset$.

As for $\tau = \infty$, clearly for every real t

$$\{\omega\colon\, \infty \leq t\} = \emptyset. \tag{25.3}$$

Indeed we know by time t whether t_* has already come (and the time $+\infty$ definitely *has not* come by time t).

Example 25.2. Let $t_1 < t_2$ be two time points belonging to T; and let C be a subset of the space of functions x_u, $u \leq t_1$. Take

$$\tau = \begin{cases} t_1 & \text{if } (X_u,\, u \leq t_1) \in C, \\ t_2 & \text{otherwise.} \end{cases} \tag{25.4}$$

Then τ is a stopping time.

Indeed,

$$\{\tau \le t\} = \begin{cases} \emptyset & \text{for } t < t_1, \\ \{(X_u,\, u \le t_1) \in C\} & \text{for } t = t_1, \\ \{(X_u,\, u \le t) \in C_t\} & \text{for } t_1 < t < t_2, \\ \Omega & \text{for } t \ge t_2, \end{cases} \tag{25.5}$$

where the set C_t in the space of all functions x_u, $u \le t$, consists of all functions whose part up to time t_1 belongs to the set C.

Example 25.3. Let $T = \{0, 1, 2, ..., n, ...\}$, and let X_t, $t \in T$, be a process taking values in a space Sp. Let A be a subset of Sp. Then τ defined as the first time for which $X_t \in A$ (the first reaching time) is a stopping time.

Only what are we to do if X_t never reaches A? and there is *no* first moment of reaching it? Let us take, in this case, $\tau = \infty$. So the precise definition:

$$\tau = \begin{cases} \min\{t\colon\ X_t \in A\} & \text{if there are such } t, \\ +\infty & \text{if there is no such } t. \end{cases} \tag{25.6}$$

This seems to coincide with the way we would use the expression "plus infinity" in our everyday life: if we tell somebody that something will happen at time $+\infty$, in all probability we would mean that it will never occur.

Let us check that this τ is a stopping time. For $t \in T$ we have:

$$\{\tau \le t\} = \{X_0 \in A\} \cup \{X_1 \in A\} \cup ... \cup \{X_t \in A\} = \{(X_u,\, 0 \le u \le t) \in C\}, \tag{25.7}$$

where the set $C \subseteq \text{Sp}^{t+1}$ is defined as the set of all sequences $(x_0, x_1, ..., x_t)$ for which at least one element $x_i \in A$.

Example 25.4. It the same situation let us consider the *last* time that $X_t \in A$:

$$\sigma = \begin{cases} \sup\{t\colon\ X_t \in A\} & \text{if there are such } t, \\ \text{and we don't know what} & \text{if there is no such } t. \end{cases} \tag{25.8}$$

The supremum is used here rather than maximum because the set of all t's for which $X_t \in A$ may be infinite (and then the supremum is equal to $+\infty$). As for the case of no such t's, not to think of it, let us restrict ourselves to the case when $X_0(\omega) \in A$ for all $\omega \in \Omega$; also let us suppose for simplicity that the number of visits of $X_t(\omega)$ to the set A is finite for every ω.

It turns out that, in general, σ is *not* a stopping time.

Indeed, say, for $t = 3$, suppose $X_0 \in A$, $X_1 \notin A$, and X_2, $X_3 \in A$. If the process X_t never visits the set A after time $t = 3$, we have $\sigma = 3$; but if it does, $\sigma > 3$. So, observing the process X_t up to time 3 we cannot be sure whether $\sigma \le 3$ or > 3.

It may, of course, happen that because of something special about the process X_t, the random variable σ happens to *be* a stopping time; but generally speaking, no.

To be continued.