MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Lecture note 26. Stopping times, continued.

Let us continue with our examples.

Example 25.3′. $T = [0, \infty)$, X_t, $t \geq 0$, is a process with continuous trajectories, and A is a closed subset of the space Sp. Then the random variable τ defined by (25.6) is a stopping time.

First of all, the minimum does exist if there are t's with $X_t \in A$: because of the continuity of $X_t(\omega)$ and closedness of the set A. If the set A were not closed, say, an open set, we could speak only of the infimum $\inf\{t\colon X_t \in A\}$.

For $t \in [0, \infty)$ we have:

$$\{\tau \leq t\} = \{(X_u,\, 0 \leq u \leq t) \in C\}, \tag{26.1}$$

where the set C consists of all functions x_u, $0 \leq u \leq t$, taking a value in the set A for at least one $u \in [0, t]$ (if we have observed X_u for $u \in [0, t]$, we just look at this function: if it reaches the set A for one of these u's, the time τ has come by the time t; if not, it hasn't).

Example 25.3″. $T = [0, \infty)$, X_t, $t \in [0, \infty)$, is a real-valued process with continuous trajectories, $A = (0, \infty)$ (an open, not closed set). Take

$$\tau = \begin{cases} \inf\{t\colon X_t \in A\} & \text{if there are such } t, \\ +\infty & \text{if there is no such } t. \end{cases} \tag{26.2}$$

It turns out that, generally, this is *not* a stopping time.

Indeed, suppose we observed the process X_t for $0 \leq t \leq 2$, and got the following realization:

$$X_t(\omega) = \begin{cases} -2 + t^2, & 0 \leq t \leq 1, \\ -(2 - t)^2, & 1 \leq t \leq 2 \end{cases} \tag{26.3}$$

(make a picture of the graph, which is continuous; at $t = 1$ both formulas yield the same result). It may be that for this ω the trajectory will go up after $t = 2$, e. g.:

$$X_t(\omega) = (t - 2)^2, \qquad t \geq 2; \tag{26.4}$$

for such an ω we have $\tau(\omega) = \inf\{t\colon t \in (2, \infty)\} = 2$. But it may be that the trajectory will go down after having touched the level 0, e. g.:

$$X_t(\omega) = -(t - 2)^2 + (t - 2)^3, \qquad t \geq 2 \tag{26.5}$$

(make a picture of the graph). For ω for which (26.3), (26.5) hold, $\tau = 3$.

So observing X_u, $0 \leq u \leq 2$, we cannot decide whether the event $\{\tau \leq t\}$ has occurred or not.

Not a stopping time.

By the way, in our example (26.3) we had $X_t(\omega) \le 0$ for $0 \le t \le 2$, the equality reached at the point 2. If $X_t(\omega)$ is < 0 for all $t \in [0, 2]$, including the last point, we can guarantee that the time $\tau(\omega)$ hasn't come yet by $t = 2$.

Example 26.1. Let τ be an arbitrary stopping time; let us prove that $\sigma = \tau + 1$ is also one (we suppose that the time parameter set T is such that $t + 1 \in T$ for $t \in T$). Let, for definiteness, $T = [0, \infty)$ or $T = \{0, 1, 2, ..., n, ...\}$.

We have:

$$\{\sigma \le t\} = \{\tau \le t - 1\} = \begin{cases} \emptyset & \text{if } t < 1, \\ \{(X_u,\, 0 \le u \le t) \in D\} & \text{if } t \ge 1, \end{cases} \tag{26.6}$$

where the set D in the space of functions x_u, $0 \le u \le t$, consists of all functions whose restriction to the interval $[0,\, t - 1]$ belongs to the set C used in the representation

$$\{\tau \le t - 1\} = \{(X_u,\, 0 \le u \le t - 1) \in C\}. \tag{26.7}$$

Example 26.1′. Let $T = [0, \infty)$, and let τ be a stopping time. The random variable $\sigma = \tau/2$ may not be a stopping time.

This is because the event

$$\{\sigma \le t\} = \{\tau \le 2t\} = \{(X_u,\, 0 \le u \le 2t) \in C_{2t}\} \tag{26.8}$$

may be representable through the values of X_u with $0 \le u \le \boldsymbol{t}$, and may be not: one cannot require you to stop the process at the half-time before it, say, reaches a set A for the first time.

Example 26.1″. Let τ be an arbitrary stopping time; and let $h(t)$ be a nondecreasing left-continuous function on T such that $h(t) \ge t$ for every t (make a picture of the graph of such a function). Then the random variable

$$\sigma = h(\tau) \tag{26.9}$$

(we take $h(\infty) = \infty$) is a stopping time.

Indeed,

$$\{\sigma \le t\} = \{\tau \le t_*\}, \tag{26.10}$$

where t_* is the largest value of $s \le t$ for which $h(s) \le t$ if there are such values:

$$t_* = \max\{s \le t\colon\, h(s) \le t\}; \tag{26.11}$$

the maximum *is* reached because of our restrictions imposed on the function $h(s)$: the set $\{s \le t\colon\, h(s) \le t\}$ is an *interval* containing its right end. If there are no values $s \le t$ for which $h(s) \le t$, the event $\{\sigma \le t\}$ is just impossible (equal to $\emptyset$).

This is a generalization of Example 26.1, not of Example 26.1′.

Example 26.2. If τ, σ are stopping times, then so are $\min(\tau,\, \sigma)$ and $\max(\tau,\, \sigma)$.

Indeed,

$$\{\min(\tau,\, \sigma) \le t\} = \{\tau \le t\} \cup \{\sigma \le t\}, \qquad \{\max(\tau,\, \sigma) \le t\} = \{\tau \le t\} \cap \{\sigma \le t\}. \tag{26.12}$$