Lecture note 28. More theory of martingales. Continuous time.

Microtheorem 28.1. *Let* Y_t, $t \in T$, *be a nonnegative submartingale with sample functions that are continuous from the right* (or from the left). *Then for every* $a > 0$

$$P\{\sup_{t\in T} Y_t \geq a\} \leq \frac{\sup_{t\in T} E(Y_t)}{a}. \tag{28.1}$$

This is a continuous-time counterpart of Microtheorem 27.4.

Proof. Clearly there exists a sequence of finite subsets $\mathfrak{T}_1 \subseteq \mathfrak{T}_2 \subseteq ... \subseteq \mathfrak{T}_n \subseteq ... \subseteq T$ such that

$$\sup_{t\in T} Y_t = \lim_{n\to\infty} \max_{t\in\mathfrak{T}_n} Y_t. \tag{28.2}$$

It would seem that

$$\{\sup_{t\in T} Y_t \geq a\} = \bigcup_{n=1}^{\infty} \{\max_{t\in\mathfrak{T}_n} Y_t \geq a\}; \tag{28.3}$$

but this is not true: it is possible that all $\max_{t\in\mathfrak{T}_n} Y_t$ are less than a, and $\sup_{t\in T} Y_t = a\}$. What is true, however, is that for every $\varepsilon \in (0, a)$

$$\{\sup_{t\in T} Y_t \geq a\} \subseteq \bigcup_{n=1}^{\infty} \{\max_{t\in\mathfrak{T}_n} Y_t \geq a - \varepsilon\}. \tag{28.4}$$

So we have (using Microtheorem 27.4):

$$\begin{aligned} P\{\sup_{t\in T} Y_t \geq a\} &\leq P\Big(\bigcup_{n=1}^{\infty} \{\max_{t\in\mathfrak{T}_n} Y_t \geq a - \varepsilon\}\Big) = \lim_{n\to\infty} P\{\max_{t\in\mathfrak{T}_n} Y_t \geq a - \varepsilon\} \\ &\leq \frac{\lim_{n\to\infty} \max_{t\in\mathfrak{T}_n} E(Y_t)}{a - \varepsilon} = \frac{\sup_{t\in T} E(Y_t)}{a - \varepsilon}. \end{aligned} \tag{28.5}$$

Since the positive ε is arbitrary, we come to (28.1).

Now we can return to the question of stochastic integrals being almost surely continuous in t (Lecture note 6). We saw that stochastic integrals of *step* random functions are continuous in t not only almost surely, but for *all* $\omega \in \Omega$. For a random function $g(t, \omega)$ let us choose a sequence of partitions $\mathfrak{T}_n$ of the half-line $t \geq t_0$ so that

$$\int_{t_0}^{t} E\big(\big(g(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)^2\big)\, du < \frac{1}{10^n} \tag{28.6}$$

for $t \in [t_0, t_{\max}]$. Then

$$\int_{t_0}^{t} E\big(\big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)^2\big)\, du < \frac{2}{10^n} + \frac{2}{10^{n+1}} < \frac{3}{10^n}. \tag{28.7}$$

The stochastic integral
$$\int_{t_0}^{t} \big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)\ dW_u, \tag{28.8}$$
being a stochastic integral of a step random function, is continuous in t for all ω; so by Microtheorem 28.1 applied to the nonnegative supermartingale
$$\Big(\int_{t_0}^{t} \big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)\ dW_u\Big)^2, \tag{28.9}$$
we have:
$$\begin{aligned} &P\Big\{\max_{t_0\le t\le t_{\max}}\Big|\int_{t_0}^{t} \big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)\ dW_u\Big| \ge \frac{1}{2^n}\Big\} \\ &\quad = P\Big\{\max_{t_0\le t\le t_{\max}}\Big(\int_{t_0}^{t} \big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)\ dW_u\Big)^2 \ge \frac{1}{4^n}\Big\} \\ &\quad \le \frac{E\Big(\Big(\int_{t_0}^{t_{\max}} \big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)\ dW_u\Big)^2\Big)}{1/4^n} \\ &\quad = \frac{\int_{t_0}^{t_{\max}} E\Big(\big(g_{\mathfrak{T}_{n+1}}(u,\omega) - g_{\mathfrak{T}_n}(u,\omega)\big)^2\Big)\ du}{1/4^n} < \frac{3}{2^n}. \end{aligned} \tag{28.10}$$
The series of these probabilities converges.

I believe I did not have the opportunity to mention

The Borel – Cantelli Lemma (the first Borel – Cantelli Lemma). *If* A_1, A_2, ..., A_n, ... *is a sequence of events, and* $\sum_{i=1}^{\infty} P(A_i) < \infty$, *then almost surely only finitely many events* A_i *occur.*

To be continued.