

Lecture note 29. More theory of martingales, continued.

I believe I did not have the opportunity to mention

**The Borel–Cantelli Lemma** (the first Borel–Cantelli Lemma). *If  $A_1, A_2, \dots, A_n, \dots$  is a sequence of events, and  $\sum_{i=1}^{\infty} P(A_i) < \infty$ , then almost surely only finitely many events  $A_i$  occur.*

This certainly does not belong to the theory of stochastic processes: just probability theory; and I am under no obligation to prove it – but here is the

**Proof.** The sequence of nonnegative random variables  $Y_n = \sum_{i=1}^n I_{A_i}$  is non-decreasing, and by the Monotone Convergence Theorem (Theorem 2.1) we have: the expected number of events  $A_i$  to occur

$$E\left(\sum_{i=1}^{\infty} I_{A_i}\right) = E\left(\lim_{n \rightarrow \infty} Y_n\right) = \lim_{n \rightarrow \infty} E(Y_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n E(I_{A_i}) = \sum_{i=1}^{\infty} P(A_i) < \infty. \quad (29.1)$$

Since the expectation is finite, the random variable is almost surely finite.

Now we apply this Lemma to the events

$$A_n = \left\{ \max_{t_0 \leq t \leq t_{\max}} \left| \int_{t_0}^t (g_{\tau_{n+1}}(u, \omega) - g_{\tau_n}(u, \omega)) dW_u \right| \geq \frac{1}{2^n} \right\}, \quad (29.2)$$

and get that almost surely for all  $n$  except finitely many

$$\left| \int_{t_0}^t (g_{\tau_{n+1}}(u, \omega) - g_{\tau_n}(u, \omega)) dW_u \right| < \frac{1}{2^n}. \quad (29.3)$$

Since the series  $\sum_{i=1}^{\infty} 1/2^i < \infty$ , we have that almost surely the series

$$\int_{t_0}^t g_{\tau_1}(u, \omega) dW_u + \sum_{i=1}^{\infty} \int_{t_0}^t (g_{\tau_{i+1}}(u, \omega) - g_{\tau_i}(u, \omega)) dW_u \quad (29.4)$$

dominated by it converges *uniformly* in  $t \in [t_0, t_{\max}]$ , which means that the sequence  $\int_{t_0}^t g_{\tau_n}(u, \omega) dW_u$  converges uniformly; and the limit of a uniformly convergent sequence of continuous functions is a continuous function. And this limit is (for every  $t$ , a version of)  $\int_{t_0}^t g(u, \omega) dW_u$ .

**Microtheorem 29.1.** *Let  $Y_t, t \in T$ , be a (sub)martingale with sample functions that are continuous on the right; let  $\tau$  be a stopping time that is bounded by some  $t_{\max} \in T$ :  $\tau \leq t_{\max}$ . Suppose all  $Y_t(\omega)$  are bounded in absolute value by the same random variable  $Z(\omega)$ ,  $E(Z) < \infty$ .*

Then

$$E(Y_\tau) = E(Y_{t_{\max}}) \quad (\text{or } \leq E(Y_{t_{\max}})). \quad (29.5)$$

Note that, while in Microtheorem 28.1 it is all the same for us whether the sample function is continuous on the right or on the left, here it is important: with right continuity this is true, and with left continuity it is not, generally. See Problems 20 – 24.

**Proof.** For definiteness, let  $T = [0, \infty)$ . For a positive  $h$ , let us define the random time  $\tau_h$  by

$$\tau_h = kh \quad \text{for } (k-1)h < \tau \leq kh, \quad (29.6)$$

$k = 0, 1, 2, \dots, n, \dots$  ( $\tau_h = 0$  if  $\tau = 0$ ). This  $\tau_h$  is a non-decreasing left-continuous function of  $\tau$ , and  $\tau_h \geq \tau$ ; so by Example 26.1'' the random variable  $\tau_h$  is also a stopping time. By Example 26.2,  $\min(\tau_h, t_{\max})$  is also a stopping time. It takes only finitely many values (not more than  $t_{\max}/h + 2$ ) and is  $\leq t_{\max}$ , so we can apply Microtheorem 27.1 or 27.2:

$$E(Y_{\min(\tau_h, t_{\max})}) = E(Y_{t_{\max}}) \quad (\text{or } \leq E(Y_{t_{\max}})). \quad (29.7)$$

By the dominated-convergence theorem (Theorem 2.2) we have:

$$E(Y_\tau) = E\left(\lim_{h \rightarrow 0^+} Y_{\min(\tau_h, t_{\max})}\right) = \lim_{h \rightarrow 0^+} E(Y_{\min(\tau_h, t_{\max})}) = E(Y_{t_{\max}}) \quad (\text{or } \leq E(Y_{t_{\max}})). \quad (29.8)$$

In fact, Microtheorem 29.1 remains true without supposing that all random variables  $Y_t$ ,  $t \in T$ , are dominated by the same random variable with finite expectation:  $|Y_t(\omega)| \leq Z(\omega)$  for all  $t, \omega$ , where  $E(Z) < \infty$ ; but the proof is much more complicated – and we really will be using the result only in the situation of domination.