Lecture note 30. Time-homogeneous diffusion processes. Dirichlet problem for elliptic differential equations.

Let us apply the theory of martingales to time-homogeneous diffusion processes; i. e. processes that are solutions of stochastic equations with coefficients not depending on the time variable:

$$d\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)\,dt + \sigma(\boldsymbol{X}_t)\;d\boldsymbol{W}_t \tag{30.1}$$

(I am writing this in the multidimensional case: $\boldsymbol{X}_t \in \mathbb{R}^r$). The differential operator corresponding to such a process does not depend on the time variable:

$$Lf(\boldsymbol{x}) = \frac{1}{2}\sum_{i,\,j=1}^{r} a_{ij}(\boldsymbol{x})\cdot\frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}) + \sum_{i=1}^{r} b_i(\boldsymbol{x})\cdot\frac{\partial f}{\partial x^i}(\boldsymbol{x}). \tag{30.2}$$

Let us speak about linear partial differential equations of the elliptic type: equations of the form

$$\mathcal{L}u(\boldsymbol{x}) = \frac{1}{2}\sum_{i,\,j=1}^{r} a_{ij}(\boldsymbol{x})\cdot\frac{\partial^2 u}{\partial x^i \partial x^j}(\boldsymbol{x}) + \sum_{i=1}^{r} b_i(\boldsymbol{x})\cdot\frac{\partial u}{\partial x^i}(\boldsymbol{x}) + c(\boldsymbol{x})\cdot u(\boldsymbol{x}) = g(\boldsymbol{x}), \tag{30.3}$$

where the matrix $\big(a_{ij}(\boldsymbol{x})\big)$ is, for every $\boldsymbol{x}$, positive definite.

For elliptic equations we consider not initial-value problems, but *boundary-value problems.*

The *Dirichlet problem* for the equation (30.3) is formulated as follows: given a function $g(\boldsymbol{x})$ in a region $G \subseteq \mathbb{R}^r$ and a function $\varphi(\boldsymbol{x})$ on its boundary ∂G, find a continuous function $u(\boldsymbol{x})$, $\boldsymbol{x} \in G \cup \partial G$, that is twice continuously differentiable in G, satisfies equation (30.3) for $\boldsymbol{x} \in G$ and satisfies the boundary condition

$$u(\boldsymbol{x}) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial G. \tag{30.4}$$

The same can be reformulated as follows: find a twice continuously differentiable function $u(\boldsymbol{x})$ in G such that

$$\begin{aligned} \mathcal{L}u(\boldsymbol{x}) &= g(\boldsymbol{x}), \qquad && \boldsymbol{x} \in G, \\ \lim_{\boldsymbol{y}\to\boldsymbol{x}} u(\boldsymbol{y}) &= \varphi(\boldsymbol{x}), && \boldsymbol{x} \in \partial G. \end{aligned} \tag{30.5}$$

For shortness, we just write the boundary condition as $u(\boldsymbol{x}) = \varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$ (but meaning that the limit condition in (30.5) is satisfied, or that the function $u(\boldsymbol{x})$ is continuous in the closed region $G \cup \partial G$).

In the case of the dimension $r = 1$, elliptic equations are second-order ordinary differential equations; and the Dirichlet problem becomes a boundary-value problem for an ordinary differential equation in an interval, with two boundary conditions at the two ends of the interval.

It turns out that the solution $u(\boldsymbol{x})$ of the Dirichlet problem

$$\begin{aligned} Lu(\boldsymbol{x}) &= g(\boldsymbol{x}), \qquad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial G, \end{aligned} \tag{30.6}$$

has the representation

$$u(\boldsymbol{x}) = E\Big(\varphi(\boldsymbol{X}_\tau^{\boldsymbol{x}}) - \int_0^\tau g(\boldsymbol{X}_s^{\boldsymbol{x}})\, ds\Big), \tag{30.7}$$

where $\boldsymbol{X}_t^{\boldsymbol{x}}$ is the diffusion process starting at time $t = 0$ from the point $\boldsymbol{x} \in G$, and $\tau = \tau_G^{\boldsymbol{x}}$ is the first time at which the process $\boldsymbol{X}_t^{\boldsymbol{x}}$ leaves the region G:

$$\tau = \tau_G^{\boldsymbol{x}} = \begin{cases} \min\{t\colon\ \boldsymbol{X}_t^{\boldsymbol{x}} \notin G\} & \text{if there are such } t, \\ \infty & \text{otherwise.} \end{cases} \tag{30.8}$$

Particular cases:

1) $g(\boldsymbol{x}) \equiv 0$: the solution $u(\boldsymbol{x})$ of the Dirichlet problem

$$\begin{aligned} Lu(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), \quad x \in \partial G, \end{aligned} \tag{30.9}$$

is represented as

$$u(\boldsymbol{x}) = E\big(\varphi(\boldsymbol{X}_\tau^{\boldsymbol{x}})\big); \tag{30.10}$$

2) $g(\boldsymbol{x}) \equiv -1$, $\varphi(\boldsymbol{x}) \equiv 0$ for $\boldsymbol{x} \in \partial G$: the expectation

$$m(\boldsymbol{x}) = E(\tau_G^{\boldsymbol{x}}) \tag{30.11}$$

is the solution of the Dirichlet problem

$$\begin{aligned} Lm(\boldsymbol{x}) &= -1, \quad \boldsymbol{x} \in G, \\ m(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \partial G. \end{aligned} \tag{30.12}$$

However, this is not as simple as that: for (30.7), (30.10), or (30.12) to hold it is necessary that $E(\tau_G^{\boldsymbol{x}})$ be finite; or at least that $\tau = \tau_G^{\boldsymbol{x}} < \infty$ almost surely (otherwise $\boldsymbol{X}_\tau^{\boldsymbol{x}}$ makes no sense). And we were speaking of *the* solution of the Dirichlet problem; but does such a thing exist: do we know that this solution is *unique*? Also questions of *existence* of a solution arise. If $E(\tau_G^{\boldsymbol{x}}) < \infty$, and the functions g and φ are bounded, the expectation (30.7) does exist; but will the function $u(\boldsymbol{x})$ defined by this formula be a solution of (30.6); will it be smooth inside G? will it assume the boundary values $\varphi(\boldsymbol{x})$ at the boundary ∂G?

These questions are best addressed with some combination of usual partial-differential-equations methods and probabilistic methods based on martingales.

And it is clear that we need some order.