MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Lecture note 31. Application of the theory of martingales to time-homogeneous diffusion processes.

It is clear that we need some order.

Microtheorem 31.1. *Let* $m(\boldsymbol{x})$ *be a solution of the Dirichlet problem* (30.12). *Suppose that* $m(\boldsymbol{x})$*, defined, originally, only for* $\boldsymbol{x} \in G \cup \partial G$*, can be extended to the whole* $\mathbb{R}^r$ *as a bounded twice continuously differentiable function* $u(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^r$ ($u(\boldsymbol{x}) = m(\boldsymbol{x})$ for $\boldsymbol{x} \in G \cup \partial G$)*, such that the function* $Lu(\boldsymbol{x})$ *and the product* $\big(\nabla u(\boldsymbol{x})\big)^T \sigma(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^r$*, are bounded.*

Then $E(\tau_G^{\boldsymbol{x}}) < \infty$ *for every* $\boldsymbol{x} \in G$*, and* $m(\boldsymbol{x}) = E(\tau_G^{\boldsymbol{x}})$.

Proof. By Itô's formula,

$$Y_t = u(\boldsymbol{X}_t^{\boldsymbol{x}}) - \int_0^t Lu(X_s^{\boldsymbol{x}})\, ds = \int_0^t \sum_{i=1}^r \sum_{k=1}^n \frac{\partial u}{\partial x^i}(X_s^{\boldsymbol{x}}) \cdot \sigma_{ik}(X_s^{\boldsymbol{x}})\, dW_s^k \tag{31.1}$$

is a martingale. The random variable $\tau = \tau_G^{\boldsymbol{x}}$ is a stopping time (the first time that a continuous stochastic process $X_t^{\boldsymbol{x}}$ hits the closed set $\mathbb{R}^r \setminus G$); and for every $t_* \in (0, \infty)$ the minimum $\min(\tau, t_*)$ is a *bounded* stopping time (bounded by the constant t_*). So by Microtheorem 29.1,

$$E(Y_{\min(\tau_G^{\boldsymbol{x}},\, t_*)}) = E(Y_0); \tag{31.2}$$

or, by (31.1),

$$E\Big(u(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) - \int_0^{\min(\tau,\, t_*)} Lu(X_s^{\boldsymbol{x}})\, ds\Big) = E\big(u(\boldsymbol{X}_0^{\boldsymbol{x}})\big) = u(\boldsymbol{x}). \tag{31.3}$$

We have $u(\boldsymbol{x}) = m(\boldsymbol{x})$ because $\boldsymbol{x} \in G$; since $\min(\tau,\, t_*) \le \tau$, we have $u(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) = m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})$ and $Lu(X_s^{\boldsymbol{x}}) = Lm(X_s^{\boldsymbol{x}}) = -1$ for $s < \min(\tau,\, t_*)$; so

$$m(\boldsymbol{x}) = E\big(m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) + \min(\tau,\, t_*)\big), \tag{31.4}$$

$$E\big(\min(\tau,\, t_*)\big) = m(\boldsymbol{x}) - E\big(m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})\big). \tag{31.5}$$

It is clear that $\min(\tau,\, t_*)$ for every $\omega \in \Omega$ is a non-decreasing function of t_* (it is *increasing* up to $t_* = \tau_G^{\boldsymbol{x}}$, and after that it remains constant; if $\tau_G^{\boldsymbol{x}} = \infty$, this minimum is equal to t_*, and keeps on growing for all t_*). Clearly the limit

$$\lim_{t_* \to \infty} \min(\tau_G^{\boldsymbol{x}},\, t_*) = \tau_G^{\boldsymbol{x}}. \tag{31.6}$$

By the monotone-convergence theorem (Theorem 2.1) we have:

$$E(\tau_G^{\boldsymbol{x}}) = \lim_{t_* \to \infty} E\big(\min(\tau_G^{\boldsymbol{x}},\, t_*)\big) = m(\boldsymbol{x}) - \lim_{t_* \to \infty} E\big(m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})\big). \tag{31.7}$$

The function m (being a restriction of a bounded function u) is bounded, so the limit (31.7) is finite, $E(\tau_G^{\boldsymbol{x}}) < \infty$.

Now, using the fact that $\tau_G^{\boldsymbol{x}} < \infty$ almost surely, and the fact that $\boldsymbol{X}_t^{\boldsymbol{x}}$ is continuous, we have:

$$\lim_{t_* \to \infty} \boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)} = \boldsymbol{X}^{\boldsymbol{x}}_{\tau}; \tag{31.8}$$

and using the fact that the function m is continuous in $G \cup \partial G$, we get:

$$\lim_{t_* \to \infty} m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) = m(\boldsymbol{X}^{\boldsymbol{x}}_{\tau}) = 0, \tag{31.9}$$

because $\boldsymbol{X}^{\boldsymbol{x}}_{\tau} \in \partial G$.

The random variables $m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})$ are all bounded by the same constant $\sup\{|m(\boldsymbol{x})|: \boldsymbol{x} \in G \cup \partial G\}$ ($\le \sup\{|u(\boldsymbol{x})|: \boldsymbol{x} \in \mathbb{R}^r\} < \infty$); so by the dominated-convergence theorem (Theorem 2.2) we have:

$$E(\tau_G^{\boldsymbol{x}}) = m(\boldsymbol{x}) - \lim_{t_* \to \infty} E\big(m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})\big) = m(\boldsymbol{x}) - E\big(\lim_{t_* \to \infty} m(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)})\big) = m(\boldsymbol{x}). \tag{31.10}$$

Microtheorem 31.2. *Let G be a region in $\mathbb{R}^r$ such that $E(\tau_G^{\boldsymbol{x}}) < \infty$. Let $g(\boldsymbol{x})$, $\boldsymbol{x} \in G$, and $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$, be bounded functions. Suppose $u(\boldsymbol{x})$ is a solution of the Dirichlet problem*

$$\begin{aligned} Lu(\boldsymbol{x}) = \frac{1}{2} \sum_{i,\, j=1}^{n} a_{ij}(\boldsymbol{x}) \cdot \frac{\partial^2 u}{\partial x^i \partial x^j}(\boldsymbol{x}) + \sum_{i=1}^{n} b_i(\boldsymbol{x}) \cdot \frac{\partial u}{\partial x^i}(\boldsymbol{x}) = g(\boldsymbol{x}), &\quad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial G. & \end{aligned} \tag{31.11}$$

Suppose $u(\boldsymbol{x})$ can be extended as a twice continuously differentiable function, bounded with its derivatives, to the whole $\mathbb{R}^r$.

Then

$$u(\boldsymbol{x}) = E\Big(\varphi(X^{\boldsymbol{x}}_{\tau}) - \int_0^{\tau} g(X^{\boldsymbol{x}}_s)\, ds\Big). \tag{31.12}$$

Proof. Again take the martingale

$$Y_t = u(\boldsymbol{X}^{\boldsymbol{x}}_t) - \int_0^t Lu(\boldsymbol{X}^{\boldsymbol{x}}_s)\, ds, \tag{31.13}$$

and again take it at the stopping time $\min(\tau,\, t_*)$:

$$\begin{aligned} &E\Big(u(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) - \int_0^{\min(\tau,\, t_*)} Lu(\boldsymbol{X}^{\boldsymbol{x}}_s)\, ds\Big) \\ &\qquad = E\Big(u(\boldsymbol{X}^{\boldsymbol{x}}_{\min(\tau,\, t_*)}) - \int_0^{\min(\tau,\, t_*)} g(\boldsymbol{X}^{\boldsymbol{x}}_s)\, ds\Big) = E(Y_0) = u(\boldsymbol{x}). \end{aligned} \tag{31.14}$$

The random variables under the expectation sign are for every t_* dominated by the random variable

$$\sup_{\boldsymbol{x}} |u(\boldsymbol{x})| + \sup_{\boldsymbol{x}} |g(\boldsymbol{x})| \cdot \tau, \tag{31.15}$$

which has a finite expectation; and by the dominated-convergence theorem, letting $t_* \to \infty$, we get

$$u(\boldsymbol{x}) = E\Big(u(\boldsymbol{X}_\tau^{\boldsymbol{x}}) - \int_0^\tau g(\boldsymbol{X}_s^{\boldsymbol{x}})\, ds\Big) = E\Big(\varphi(\boldsymbol{X}_\tau^{\boldsymbol{x}}) - \int_0^\tau g(\boldsymbol{X}_s^{\boldsymbol{x}})\, ds\Big). \tag{31.16}$$

Example 31.1. Let W_t^x be a (one-dimensional) Wiener process starting from a point x at time 0; let (a, b) be an interval containing the point x. If we want to find the expectation $E(\tau_{(a,\, b)}^x)$ (and to find out whether this expectation is finite), the way to do it is to find this expectation (denote it $m(x)$) at once for *all* points $x \in (a, b)$ (no problem finding this expectation for $x = a$ or b: it is equal to 0, because $\tau_{(a,\, b)}^a = \tau_{(a,\, b)}^a = 0$). We look for $m(x)$ as the solution of the boundary-value problem

$$\begin{aligned} &\frac{1}{2}\, m''(x) = -1, \qquad a < x < b, \\ &m(a) = m(b) = 0. \end{aligned} \tag{31.17}$$

The solution is easily found:

$$m(x) = (b - x)(x - a), \qquad a \le x \le b \tag{31.18}$$

(check that this is a solution).

The formula (31.18) defines a function that is defined on the whole real line; but it is not bounded. So we change the function outside the interval $[a, b]$, taking, say,

$$u(x) = (b - x)(x - a) \cdot h(x - b) \cdot h(a - x), \tag{31.19}$$

where

$$h(x) = \begin{cases} 1, & x \le 0, \\ 1 - 10x^3 + 15x^4 - 6x^5, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases} \tag{31.20}$$

The function u is bounded everywhere with its first two derivatives, and it coincides with m in the interval $[a, b]$. So by Microtheorem 31.1 the expectation $E(\tau_{(a,\, b)}^x)$ is given by (31.18).

So $\tau = \tau_{(a,\, b)}^x$ is finite almost surely, and we can speak of the distribution of the exit point W_τ^x: the probabilities $P\{W_\tau^x = a\}$, $P\{W_\tau^x = b\}$.

By Microtheorem 31.2, we look for the expectation $u(x) = E\big(\varphi(W_\tau^x)\big)$ as the solution of the boundary-value problem

$$\begin{aligned} &\frac{1}{2}\, u''(x) = 0, \qquad x \in (a, b), \\ &u(a) = \varphi(a), \quad u(b) = \varphi(b). \end{aligned} \tag{31.21}$$

The second derivative is equal to 0: the function is linear. It is found easily:

$$u(x) = \varphi(a) \cdot \frac{b-x}{b-a} + \varphi(b) \cdot \frac{x-a}{b-a}, \qquad a \le x \le b \tag{31.22}$$

(check it). Again a function is defined by this formula in the whole real line – smooth, but not bounded; again it can be helped the same way. So the expectation $u(x) = E\bigl(\varphi(W_\tau^x)\bigr)$ is given by (31.22). On the other hand,

$$E\bigl(\varphi(W_\tau^x)\bigr) = \varphi(a) \cdot P\{W_\tau^x = a\} + \varphi(b) \cdot P\{W_\tau^x = b\}. \tag{31.23}$$

Using the fact that $P\{W_\tau^x = a\} + P\{W_\tau^x = b\} = 1$, we get one linear algebraic equation with one unknown, say, $P\{W_\tau^x = b\}$, which has a unique solution. So we get:

$$P\{W_\tau^x = a\} = \frac{b-x}{b-a}, \quad P\{W_\tau^x = b\} = \frac{x-a}{b-a}, \qquad a \le x \le b. \tag{31.24}$$

This is the general situation: if we can solve the problem (30.9) for a sufficiently wide class of boundary functions φ, we find the distribution of the boundary point $\boldsymbol{X}_\tau^{\boldsymbol{x}}$ at which the process $\boldsymbol{X}_t^{\boldsymbol{x}}$ leaves the region G.

We can also consider the times $\tau_{(a,\,\infty)}^x$, $\tau_{(-\infty,\,b)}^x$ of leaving an infinite interval containing the initial point x. Are these times almost surely finite? (If they are, there is no question about $W_{\tau_{(a,\,\infty)}^x}^x$, $W_{\tau_{(-\infty,\,b)}^x}^x$: the process can leave a semi-infinite interval only through its only end.)

We have for all $\omega \in \Omega$:

$$\tau_{(a,\,\infty)}^x = \lim_{b \to \infty} \tau_{(a,\,b)}^x. \tag{31.25}$$

As for the event $\{\tau_{(a,\,\infty)}^x < \infty\}$, it is represented as

$$\{\tau_{(a,\,\infty)}^x < \infty\} = \bigcup_{b > x} \{W_{\tau_{(a,\,b)}^x}^x = a\}. \tag{31.26}$$

Indeed, if for some $\omega \in \Omega$ the event in the left-hand side occurs, W_t^x leaves the interval (a, b) at the time $\tau_{(a,\,\infty)}^x$ through its left end (make a picture). The maximum $\max_{0 \le t \le \tau_{(a,\,\infty)}^x} W_t^x$ is finite, and for b greater than this maximum the event $\{W_{\tau_{(a,\,b)}^x}^x = a\}$ occurs. So $\{\tau_{(a,\,\infty)}^x < \infty\} \subseteq \bigcup_{b>x} \{W_{\tau_{(a,\,b)}^x}^x = a\}$.

The opposite inclusion: if the event in the right-hand side of (31.26) occurs, then for some $b \ge x$ the event $\{W_{\tau_{(a,\,b)}^x}^x = a\}$ occurs; and at some time the Wiener process *is* at the point a: we have left the interval (a, ∞).

The family of events in the right-hand side of (31.25) is non-decreasing ($\{W_{\tau_{(a,\,b_1)}^x}^x = a\} \subseteq \{W_{\tau_{(a,\,b_2)}^x}^x = a\}$ for $b_2 > b_1$), so

$$P\{\tau_{(a,\,\infty)}^x < \infty\} = \lim_{b \to \infty} P\{W_{\tau_{(a,\,b)}^x}^x = a\} = \lim_{b \to \infty} \frac{b-x}{b-a} = 1. \tag{31.27}$$

So the Wiener process *does* leave almost surely every semi-finite interval, it does almost surely reach every point a to the left of the starting point x, and every point b to the right of it. It follows from this that almost surely

$$\varliminf_{t\to\infty} W_t^x = -\infty, \qquad \varlimsup_{t\to\infty} W_t^x = \infty \tag{31.28}$$

(make a picture of a function defined for $t \in [0, \infty)$ and oscillating wider and wider as $t \to \infty$).

So almost surely $\tau^x_{(a,\,\infty)} < \infty$, $\tau^x_{(-\infty,\,b)} < \infty$. What about the expectations of these random variables?

From (31.25), using the monotone-convergence theorem, we get:

$$E(\tau^x_{(a,\,\infty)}) = \lim_{b\to\infty} E(\tau^x_{(a,\,b)}) = \lim_{b\to\infty} (b-x)(x-a) = \infty. \tag{31.29}$$

When we started, in the elementary probability theory course, with expectations, we were shown some examples of random variables having no finite expectations; these examples seemed to be pretty artificial. But here we have quite a natural example of a random variable $\tau = \tau^x_{(a,\,\infty)}$ having infinite expectation.