Lecture note 32. Time-homogeneous diffusion processes, continued.

We were able to find the solution $m(x)$ of the boundary-value problem (31.17) because it was one for an *ordinary* linear differential equation. Not always will we be so lucky. Let us have a result that we'll be able to use in a wider class of situations.

Microtheorem 31.1′. *Let G be an open region, bounded or not, in $\mathbb{R}^r$; let $\tau = \tau_G^{\boldsymbol{x}}$ be the first time at which the process $\boldsymbol{X}_t^{\boldsymbol{x}}$ starting from the point $\boldsymbol{x}$ at time $t = 0$ leaves G ($\tau = \infty$ if $\boldsymbol{X}_t$ never leaves G at all). Suppose $u(\boldsymbol{x})$ is a bounded twice continuously differentiable function with bounded $\big(\sigma(\boldsymbol{x})\big)^T \nabla u(\boldsymbol{x})$ in $\mathbb{R}^r$; suppose*

$$Lu(\boldsymbol{x}) \le -\,c_0 = \text{const} < 0, \qquad \boldsymbol{x} \in G. \tag{32.1}$$

Then the expectation $E(\tau_G^{\boldsymbol{x}})$ is finite, moreover,

$$E(\tau_G^{\boldsymbol{x}}) \le \frac{2\sup_{\boldsymbol{x}} u(\boldsymbol{x})}{c_0}. \tag{32.2}$$

Proof. The random function

$$Y_t = u(\boldsymbol{X}_t^{\boldsymbol{x}}) - \int_0^t Lu(\boldsymbol{X}_s^{\boldsymbol{x}})\, ds \tag{32.3}$$

is a martingale, because by Itô's formula it is a stochastic integral. Clearly $\tau_G^{\boldsymbol{x}}$ is a stopping time, and so is the bounded random variable $\min(\tau_G^{\boldsymbol{x}}, t_*)$ for every non-random $t_* \in (0, \infty)$. So we have:

$$E(Y_{\min(\tau_G^{\boldsymbol{x}},\, t_*)}) = E(Y_0) = u(\boldsymbol{x}), \tag{32.4}$$

$$\begin{aligned} E\big(\min(\tau_G^{\boldsymbol{x}},\, t_*)\big) &\le \frac{1}{c_0} E\Big(-\int_0^{\min(\tau_G^{\boldsymbol{x}},\, t_*)} Lu(X_s^{\boldsymbol{x}})\, ds\Big) \\ &= \frac{u(\boldsymbol{x}) - u(X^{\boldsymbol{x}}_{\min(\tau_G^{\boldsymbol{x}},\, t_*)})}{c_0} \le \frac{2\sup_{\boldsymbol{x}} u(\boldsymbol{x})}{c_0}. \end{aligned} \tag{32.5}$$

Now we take $t_* \to \infty$ and by the monotone-convergence theorem obtain (32.2).

Microtheorem 32.1. *Let $\boldsymbol{X}_t$ be a diffusion process with bounded coefficients $\boldsymbol{b}(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$. Let G be a region that is bounded at least in the direction of some coordinate x^i:*

$$\sup\{|x^i| \colon \boldsymbol{x} = (x^1, ..., x^r) \in G\} < \infty. \tag{32.6}$$

Let the diffusion coefficient in the direction of this coordinate be bounded from below by a positive constant in G:

$$a_{ii}(\boldsymbol{x}) \ge a_0 > 0, \qquad \boldsymbol{x} \in G. \tag{32.7}$$

Then $E(\tau_G^{\boldsymbol{x}}) < \infty$.

Proof. Take

$$f(\boldsymbol{x}) = f(x^1, ..., x^r) = -\cosh(Cx^i) \tag{32.8}$$

(the hyperbolic cosine). If we take $C = 2\sup_x |\boldsymbol{b}(\boldsymbol{x})|/a_0$, we have:

$$Lf(\boldsymbol{x}) \le -c_0 = -\big(\sup_x |\boldsymbol{b}(\boldsymbol{x})|\big)/a_0 < 0, \qquad \boldsymbol{x} \in \mathbb{R}^r. \tag{32.9}$$

However the function f is unbounded. This can be helped: take

$$u(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot h\big(|x^i| - \sup\{|x^i|\colon \boldsymbol{x} \in G\}\big), \tag{32.10}$$

where $h(x)$ is the function defined by formula (31.20). The function u is bounded everywhere with its first two derivatives, and it coincides with f in the region G together with its boundary.

Microtheorem 31.1′ yields our statement.

Example 32.1. Let $X_t = W_t + b \cdot t$, $b = \text{const} \ne 0$. This is a diffusion process with drift identically equal to b, and the diffusion coefficient 1. For $G = (a, c)$ the time $\tau = \tau^x_{(a,\,c)}$ has a finite expectation for every $x \in [a, c]$ (by Microtheorem 32.1); so we can use Microtheorem 31.2. The solution of the problem

$$\begin{aligned} &Lu(x) = \frac{1}{2}\, u''(x) + b\, u'(x) = 0, \qquad a < x < b, \\ &u(a) = 0, \quad u(c) = 1 \end{aligned} \tag{32.11}$$

with $\varphi(a) = 0$, $\varphi(c) = 1$ is given by

$$u(x) = \frac{e^{-2bx} - e^{-2ba}}{e^{-2bc} - e^{-2ba}}. \tag{32.12}$$

This formula defines a smooth function everywhere, not only in the interval $[a,\, c]$; and it can be changed outside this interval so that it remains smooth, and is equal to 0 outside the interval $(a - 1,\, c + 1)$. So we have:

$$E\big(\varphi(X^x_\tau)\big) = P\{X^x_\tau = c\} = \frac{e^{-2bx} - e^{-2ba}}{e^{-2bc} - e^{-2ba}}. \tag{32.13}$$

This is the probability of leaving the interval $(a,\, c)$ through its right end; the opposite probability

$$P\{X^x_\tau = a\} = \frac{e^{-2bc} - e^{-2bx}}{e^{-2bc} - e^{-2ba}}. \tag{32.14}$$

You'll learn more about the process $W_t + b \cdot t$ solving Problems **25**–**27**.

Microtheorem 32.2. *Let G be a region in $\mathbb{R}^r$ such that $E(\tau_G^{\boldsymbol{x}}) < \infty$. Let $g(\boldsymbol{x})$, $\boldsymbol{x} \in G$, and $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$, be bounded functions. Suppose $u(\boldsymbol{x})$ is a bounded solution of the Dirichlet problem* (31.11).

Then the equality (31.12) *holds.*

That is: we can do *without* the condition of the solution extended to the whole space.

Proof. Let $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots \subset G$ be a sequence of bounded regions such that $G_n \cup \partial G_n \subset G$, and $\bigcup_{n=1}^{\infty} G_n = G$. Let us change the function $u(\boldsymbol{x})$ outside $G_n \cup \partial G_n$ so that the new function is twice continuously differentiable, and equal to 0 outside some bounded region $G_n' \supset G_n \cup \partial G_n$. Let us denote the function $u(\boldsymbol{x})$ changed so, $u_n(\boldsymbol{x})$, and let $\tau_n = \tau_{G_n}^{\boldsymbol{x}}$ be the time at which the process $\boldsymbol{X}_t^{\boldsymbol{x}}$ leaves the region G_n. By Microtheorem 31.2 we have:

$$u_n(\boldsymbol{x}) = E\Big(u_n(X_{\tau_n}^{\boldsymbol{x}}) - \int_0^{\tau_n} g(X_s^{\boldsymbol{x}})\, ds\Big). \tag{32.15}$$

Since $\boldsymbol{x} \in G_n$ (this, at least for sufficiently large n), and $X_{\tau_n}^{\boldsymbol{x}} \in G_n \cup \partial G_n$, we have:

$$u(\boldsymbol{x}) = E\Big(u(X_{\tau_n}^{\boldsymbol{x}}) - \int_0^{\tau_n} g(X_s^{\boldsymbol{x}})\, ds\Big). \tag{32.16}$$

Now, as $n \to \infty$, we have $\tau_n \to \tau$, $X_{\tau_n}^{\boldsymbol{x}} \to X_\tau^{\boldsymbol{x}} \in \partial G$, $u(X_{\tau_n}^{\boldsymbol{x}}) \to \varphi(X_\tau^{\boldsymbol{x}})$, $\displaystyle\int_0^{\tau_n} g(X_s^{\boldsymbol{x}})\, ds \to \int_0^{\tau} g(X_s^{\boldsymbol{x}})\, ds$. All random variables under the expectation sign in (32.16) are dominated in absolute value by the same random variable $\sup_{\boldsymbol{x}} |u(\boldsymbol{x})| + \sup_{\boldsymbol{x}} |g(\boldsymbol{x})| \cdot \tau_G^{\boldsymbol{x}}$, and this random variable has a finite expectation. By the dominated-convergence theorem we get (31.12).

Microtheorem 32.2′. *Let G be a region in $\mathbb{R}^r$ such that almost surely $\tau = \tau_G^{\boldsymbol{x}} < \infty$* (finiteness of the expectation $E(\tau_G^{\boldsymbol{x}})$ is not required). *Let $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$, be a bounded function. Suppose $u(\boldsymbol{x})$ is a bounded solution of the Dirichlet problem*

$$\begin{aligned} Lu(\boldsymbol{x}) &= 0, \qquad && \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), && \boldsymbol{x} \in \partial G. \end{aligned} \tag{32.17}$$

Then

$$u(\boldsymbol{x}) = E\big(\varphi(\boldsymbol{X}_\tau^{\boldsymbol{x}})\big). \tag{32.18}$$

The **proof** is the same.

From Microtheorem 32.2′ uniqueness of a bounded solution of the Dirichlet problem follows.

Note that without the condition of the solution $u(\boldsymbol{x})$ being bounded, there is no uniqueness (and of course, formulas (31.12), (32.18) do not necessarily hold):

Example 32.2. Let $\boldsymbol{W}_t = (W_t^1, W_t^2)$ be the two-dimensional Wiener process; let $G = \{\boldsymbol{x} = (x^1, x^2) \colon x^2 > 0\}$ be the upper half-plane. The differential operator corresponding to the Wiener process is one-half of the Laplace operator: $\frac{1}{2}\Delta u(\boldsymbol{x}) = \frac{1}{2}\big(\frac{\partial^2 u}{(\partial x^1)^2} + \frac{\partial^2 u}{(\partial x^2)^2}\big)$. The function $u(x^1, x^2) = x^2$ is a solution of the Dirichlet problem

$$\begin{aligned} \frac{1}{2}\Delta u &= 0, \qquad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \partial G, \end{aligned} \tag{32.19}$$

but it is not the identical zero: (32.18) does not hold.